LM04 · DERIVATIVES · CFA LEVEL I

Arbitrage, Replication & Cost of Carry in Pricing Derivatives

The law of one price, no-arbitrage pricing, and the cost-of-carry formula

What This Reading Covers

Arbitrage is the opportunity for a riskless profit. A derivative is priced so that **no arbitrage opportunities exist** (no-arbitrage pricing). **Replication** uses borrowing/lending + the underlying to construct a forward-equivalent. **Cost of carry** is the net of costs and benefits related to owning the underlying for a specific period.

Learning Outcome Statements

LOS a Explain how the concepts of arbitrage and replication are used in pricing derivatives

LOS b Explain the difference between the spot and expected future price of an underlying and the cost of carry associated with holding the underlying asset

Exam Tip: Core formulas to memorize: $F_0(T) = S_0(1+r)^T$ (discrete) or S_0e^{rT} (continuous), and with cash flows: $F_0(T) = [S_0 - PV_0(I) + PV_0(c)](1+r)^T$.

LOS A · FOUNDATIONS

Time Value of Money — the Building Blocks

Two compounding conventions

$$
\begin{aligned}
\mathrm{FV} = \mathrm{PV}(1+r)^{T} \quad - discretecompounding - usedforindividualassets \\
\mathrm{FV} = \mathrm{PV} \cdot e^{rT} \quad - continuouscompounding - usedforportfolios, equityindexes, FX
\end{aligned}
$$

r	Discount rate — a **risk-free rate**, typically a **market reference rate (MRR)**, not a government rate
Use case (discrete)	Individual commodity / bond underlying
Use case (continuous)	Equity indexes, FX, continuous-payout commodities

WORKED EXAMPLE

Gold Futures — Cost of Carry (no cash flows)

GIVEN

Gold spot @ 1,770/oz; r = 2%; T = 3 months = 0.25 yr.

REPLICATION

Borrow 1,770 for 3 mos → FV = 1,770 × $(1.02)^{0.25}$ = **1,778.78/oz**.
GC contract is 100 oz → contract value = **177,878.00**.

NO-ARBITRAGE PRICE

$F_0(T)$ = 1,778.78/oz. Any $F_0(T)$ above or below triggers arbitrage.

LOS A · ARBITRAGE RULE

The Arbitrage Rule — Buy Low, Sell High

WORKED EXAMPLE **Gold at Mis-Priced Futures**

GIVEN

Gold @ 1,770/oz; r = 2%; T = 3 mos. FV of borrowing = 1,778.78/oz. Suppose $F_0(T)$ = **1,792.13** (mispriced too high).

ARBITRAGE TRADE

Buy low: buy gold @ 1,770/oz with borrowed funds.
Sell high: sell futures @ 1,792.13.
At t = T: deliver 100 oz for 179,213; repay loan of 177,878.
Riskless profit = 179,213 − 177,878 = **1,334.56 per contract**.

NO-ARBITRAGE FORMULA

$F_0(T) = S_0(1 + r)^T$ or $F_0(T) = S_0e^{rT}$ (assuming no transaction costs, storage, transportation, or other cash inflows).

Higher r → higher $F_0(T)$. Longer T → higher $F_0(T)$. The forward price lies *above* the spot price when carry costs (r) exceed carry benefits.

LOS A · REPLICATION

Replication — Using the Cash Market to Mimic a Forward

Replication uses *borrowing/lending* combined with *buying/selling the underlying* to replicate a forward contract (or the reverse — using a forward to replicate a cash transaction).

WORKED EXAMPLE

Oil — Replicate a 6-Month Purchase

GIVEN

Need 1,000 barrels of oil in 6 months. S_0 = 108/bbl; T = 0.5 yr; r = 2%.

ROUTE A — REPLICATE VIA CASH MARKET

Borrow 108,000 @ 2% for 6 mos; buy 1,000 bbl today. In 6 months, pay back 108,000 × $(1.02)^{0.5}$ = **109,074.65**.

ROUTE B — BUY FORWARD

$F_0(T) = 108 \times (1.02)^{0.5}$ = **109.07465/bbl**. In 6 mos: take delivery, pay 109,074.65 total — same economics as Route A.

SANITY CHECK — COMPLETE HEDGE EARNS R_F

If oil is $115/bbl at t = T:

Gain on $F_0(T)$ = 115,000 − 109,074.65 = 5,925.35.

Holding 1,000 bbl inventory + sold forward at 109.07465: deliver underlying → gain 7,000 on inventory, less 1,074.65 interest = **5,925.35**. The hedged return = r_f = 2%.

LOS B · COST OF CARRY

Cost of Carry Formula

$$F_0(T) = [S_0 - PV_0(I) + PV_0(c)] \cdot (1+r)^T$$
$$or, continuously: F_0(T) = S_0 \cdot e^{(r+c-i)T}$$

r Discount / borrowing rate (carry *cost*)

I (or i) Income / benefits — dividends, coupon interest, convenience yield (carry *benefits*)

c Costs of carrying — storage, insurance, transportation

Intuition. Own the underlying → pay the cost of capital (r) + physical costs (c), receive income (i). The forward price adjusts for this net cost of carry.

- If costs > benefits → $F_0(T) > S_0$ (*contango*)
- If costs = benefits → $F_0(T) = S_0$
- If costs < benefits → $F_0(T) < S_0$ (*backwardation*)

WORKED EXAMPLE **Stock with Discrete Dividends**

GIVEN

$S_0 = 50$; r = 5%; T = 6 months. Quarterly dividend of 0.30 at T = 3 mos and T = 6 mos.

NO-DIVIDEND CASE

$F_0(T) = 50 \times (1.05)^{0.5} =$ **51.23**

WITH TWO $0.30 DIVIDENDS

PV of divs = $0.30/(1.05)^{0.25} + 0.30/(1.05)^{0.5} = 0.2964 + 0.2928$

$F_0(T) = (50 - 0.2964 - 0.2928) \times (1.05)^{0.5} =$ **50.631**

CONTINUOUS ANALOGUE (DIV YIELD 2.4%)

LN(1.05) = 0.04879; LN(1.024) = 0.023716.

$F_0(T) = 50 \times e^{(0.04879 - 0.023716) \times 0.5}$ = **50.6307** — matches the discrete result.

QUIZ 1 · ARBITRAGE & CARRY

Numeric Drills

Q1. The no-arbitrage forward price on a non-income-paying underlying with S_0 = 100, r = 4%, T = 1 year is:

Q2. With S_0 = 50, r = 5%, T = 0.5, two $0.30 dividends at Q3 and Q6, $F_0(T)$ is closest to:

Q3. Higher benefits of holding (income, convenience yield) *most likely*:

Q4. Oil replication: need 1,000 bbl in 6 mos, S_0 = 108, r = 2%. The pay-at-delivery amount via cash-market replication is closest to:

Q5. A complete hedge (long underlying + short forward at $F_0(T)$) earns:

LOS B · CURRENCIES

FX Forward — Covered Interest Parity

Spot quote: $\mathbf{S^{P/B}}$ = amount of *price* currency needed to buy 1 unit of the *base* currency.

Long FX forward = purchase the base at expiration and pay the price currency.

Example: CAD/USD = 1.2590 → buy 1 USD (base) for 1.2590 CAD (price).

No-arbitrage (covered interest parity)

$$F^{\mathrm{P}/B} = S^{\mathrm{P}/B} \cdot (1+r_{\mathrm{P}})^T/(1+r_B)^T$$
$$or, continuously: F^{\mathrm{P}/B} = S^{\mathrm{P}/B} \cdot e^{(r_{\mathrm{P}}-r_B)T}$$

- If $r_P - r_B < 0$ → $F^{P/B} < S^{P/B}$ — base trades at **discount** (price trades at premium)
- If $r_P - r_B > 0$ → $F^{P/B} > S^{P/B}$ — base trades at **premium** (price trades at discount)

Exam Tip: Quote direction matters. The forward of a currency pair adjusts so that investing in either currency, hedged, returns the same amount.

LOS B · COMMODITIES

Contango, Backwardation & Convenience Yield

Convenience yield — a non-cash benefit of holding a physical commodity rather than a derivative (e.g., ability to meet unexpected demand, supply disruptions).

CONTANGO

$F_1(T) > S_0$

- Well-supplied market
- High storage costs
- Low convenience yield
- All $F(T) > S_0$

BACKWARDATION

$F_1(T) < S_0$

- Tight spot market
- Low storage costs
- High convenience yield
- All $F(T) < S_0$

$$F_0(T) = [S_0 + PV_0(c)] \cdot (1+r)^T \ or \ F_0(T) = S_0 \cdot e^{(r-c)T}$$

WORKED EXAMPLE **Gold with Storage Cost (end-of-period)**

GIVEN

Gold @ 1,783.28/oz; r = 2%; T = 3 mos. Storage = $2/oz paid at t = T.

TWO EQUIVALENT FORMULATIONS

Method 1 (bring storage to t=0): $F_0(T) = [1{,}783.28 + 2/(1.02)^{0.25}] \times (1.02)^{0.25} =$ **1,794.13**

Method 2 (leave storage at t=T): $F_0(T) = 1{,}783.28 \times (1.02)^{0.25} + 2 =$ **1,794.13**

EXTENDED EXAMPLE **Commodity with Quarterly Costs**

GIVEN

S_0 = 1,097; r_f = 1.5%; costs = $1/oz at quarter end.

CARRY AT MULTIPLE HORIZONS

3 mos: $F_0(T) = 1{,}097 \times (1.015)^{0.25} + 1 =$ **1,102.09**

6 mos: $F_0(T) = [1{,}097 + 1/(1.015)^{0.25}] \times (1.015)^{0.5} + 1 =$ **1,107.20**

1 yr: $F_0(T) = [1{,}097 + 1/(1.015)^{0.25} + 1/(1.015)^{0.5} + 1/(1.015)^{0.75}] \times (1.015) + 1 =$ **1,117.48**

QUIZ 2 · FX & COMMODITIES

FX & Commodity Pricing

Q6. Under covered interest parity, if $r_P > r_B$ then the **base** currency forward:

Q7. A backwardated commodity futures curve is *most consistent* with:

Q8. Gold at 1,783.28/oz with r = 2%, T = 3 mos, storage $2/oz paid at T. The 3-month forward is closest to:

Q9. Convenience yield *most likely*:

Q10. In the gold-futures arbitrage with S_0 = 1,770, r = 2%, T = 0.25, $F_0(T)$ quoted = 1,792.13, the arbitrage profit *per 100-oz contract* is closest to:

QUIZ 3 · INTEGRATION

Integration Questions

Q11. Which best captures the cost-of-carry intuition?

Q12. The discrete and continuous versions of the cost-of-carry formula give *identical* answers if:

Q13. The law of one price in derivative pricing states:

Q14. Replicating a forward on an underlying with no dividends using the cash market requires:

Q15. An analyst observes $F_0(T) < S_0$. Which is the *most likely* explanation?

MODULE COMPLETE

Your Scorecard

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LM04 Key Takeaways

Arbitrage & Replication

1. **Arbitrage** = riskless profit. Derivatives are priced so that no such opportunity exists.
2. **Law of one price:** identical assets same price; assets with known future price → spot = PV at r_f.
3. **Replication:** borrow S_0 at r, buy underlying → at T owe $S_0(1+r)^T = F_0(T)$.

Cost of Carry Formulas

4. **No cash flows:** $F_0(T) = S_0(1+r)^T = S_0e^{rT}$
5. **With income I & costs c:** $F_0(T) = [S_0 - PV(I) + PV(c)](1+r)^T = S_0e^{(r + c - i)T}$
6. **FX (covered interest parity):** $F^{P/B} = S^{P/B}(1+r_P)^T / (1+r_B)^T$

Numeric Anchors

7. Gold @ 1,770, r = 2%, T = 0.25: F = 1,778.78; contract value 177,878. At F = 1,792.13 → arbitrage profit of 1,334.56/contract.
8. Oil hedge: S_0=108, T=0.5, r=2% → repay 109,074.65; at S_T=115, hedge locks in r_f.
9. 50/5%/0.30 qtr div → F = 50.631. Continuous with i=2.4% → 50.6307 (matches).

Last-Minute Checklist

No-arb: $F_0(T) = S_0(1+r)^T$ · **Full carry:** $[S - PV(I) + PV(c)](1+r)^T$

Contango: F > S (costs > benefits) · **Backwardation:** F < S (benefits > costs)

FX parity: $F^{P/B}/S^{P/B} = (1+r_P)^T/(1+r_B)^T$

Hedge earns r_f

Step 1 of 11