LM10 · DERIVATIVES · CFA LEVEL I

Valuing a Derivative Using a One-Period Binomial Model

Hedge-ratio replication and risk-neutral probabilities — two routes to the same C_0

What This Reading Covers

In a one-period binomial model, the underlying can go up (by factor R^u) or down (R^d). We build a risk-free hedge portfolio using *h* shares and a short call, equate its up- and down-state values, and discount at r to solve for C_0. The same answer comes from risk-neutral valuation using $\pi = (1+r - R^d)/(R^u - R^d)$.

Learning Outcome Statements

LOS a Explain how to value a derivative using a one-period binomial model

LOS b Describe the concept of risk neutrality in derivatives pricing

Exam Tip: Two formulas worth burning in: $\mathbf{h = (c^u - c^d)/(S^u - S^d)}$ and $\mathbf{\pi = (1+r - R^d)/(R^u - R^d)}$.

LOS A · THE TREE

The One-Period Binomial Tree

Set-up

At t=0 the stock is S_0. At t=1, the price is either:

$$S_1^u = R^u \times S_0 \ (up)$$
$$S_1^d = R^d \times S_0 \ (down)$$

Call payoffs at t=1:

$$c^u = max(0, S_1^u - X) \ \cdot \ c^d = max(0, S_1^d - X)$$

Hedge portfolio

Sell 1 call at C_0 and buy h shares: $V_0 = hS_0 - C_0$. Choose h such that $V_1^u = V_1^d$ — this makes the portfolio **risk-free** and therefore must earn r.

LOS A · SOLVE FOR H

Deriving the Hedge Ratio

Equate up- and down-state values

$$hS_1^u - c^u = hS_1^d - c^d$$

Solve for h:

$$h(S_1^u - S_1^d) = c^u - c^d$$

$$\mathbf{h = (c^u - c^d)/(S_1^u - S_1^d)}$$

Discount the hedge

$$(hS_0 - C_0)(1 + r)^T = V^u (= V^d)$$

$$\mathbf{C_0 = hS_0 - V^u/(1 + r)^T}$$

Why this works

A riskless portfolio must earn r (no-arbitrage). Equating hedged values at both nodes eliminates uncertainty, and the current value of the portfolio is its end-period value discounted at r.

LOS A · WORKED EXAMPLE

MarkMeldrum Example — S_0=80, X=100

WORKED EXAMPLE 1

Solve for C_0 by hedge ratio

GIVEN

$S_0 = 80$, $X = 100$, $R^u = 1.375$, $R^d = 0.75$, $r = 5\%$, $T = 1$

STEP 1 — NODE PRICES

$S_1^u = 80 \times 1.375 = \mathbf{110}$ $S_1^d = 80 \times 0.75 = \mathbf{60}$

STEP 2 — CALL PAYOFFS

$c^u = \max(0, 110 - 100) = \mathbf{10}$ $c^d = \max(0, 60 - 100) = \mathbf{0}$

STEP 3 — HEDGE RATIO

$h = (10 - 0)/(110 - 60) = 10/50 = \mathbf{0.2}$

STEP 4 — HEDGED PORTFOLIO VALUE AT T=1

$V^u = 0.2(110) - 10 = \mathbf{12}$ $V^d = 0.2(60) - 0 = \mathbf{12}$ ✓

STEP 5 — DISCOUNT & SOLVE FOR C_0

$C_0 = 0.2(80) - 12/1.05 = 16 - 11.42857$

ANSWER

$\mathbf{C_0 = 4.57}$

LOS A · CURRICULUM EXAMPLE #1

IFT Example — S_0=50, X=55

WORKED EXAMPLE 2 **Hightest Capital — call with 20% up/down**

GIVEN

$S_0 = 50$, $X = 55$, $R^u = 1.20$, $R^d = 0.80$, $r = 5\%$, $T = 1$ year

STEP 1 — NODE PRICES & PAYOFFS

$S_1^u = 50 \times 1.2 = 60$, $c^u = \max(0, 60 - 55) = \mathbf{5}$

$S_1^d = 50 \times 0.8 = 40$, $c^d = \max(0, 40 - 55) = \mathbf{0}$

STEP 2 — HEDGE RATIO

$h = (5 - 0)/(60 - 40) = 5/20 = \mathbf{0.25}$

STEP 3 — V^U

$V^u = 0.25(60) - 5 = \mathbf{10}$

STEP 4 — PV & CALL VALUE

$V_0 = 10/1.05 = 9.5238$

$C_0 = 0.25(50) - 9.5238 = 12.50 - 9.5238$

ANSWER

$\mathbf{C_0 \approx 2.98}$ (MM shows 2.976)

QUIZ 1 · HEDGE-RATIO REPLICATION

Check Your Understanding — Quiz 1

Q1. The hedge ratio for a call in the binomial model is:

Q2. With S_0=80, R^u=1.375, R^d=0.75, X=100: h equals:

Q3. Using the MM setup above with r = 5%, C_0 equals:

Q4. In the IFT example (S=50, X=55, R^u=1.2, R^d=0.8, r=5%): C_0 equals:

Q5. The hedged portfolio in the binomial model must earn:

LOS A · VOLATILITY EFFECT

Higher Volatility Raises the Call Value

WORKED EXAMPLE 3 **IFT Example #1 continued — R^u=1.40, R^d=0.60**

GIVEN

Same S_0=50, X=55, r=5% but now R^u = 1.40, R^d = 0.60 (wider up/down = higher vol).

STEP 1

S^u = 70, c^u = 70 − 55 = **15**

S^d = 30, c^d = **0**

STEP 2 — H

$h = (15 - 0)/(70 - 30) = 15/40 = \mathbf{0.375}$

STEP 3 — V^U

$V^u = 0.375(70) - 15 = 26.25 - 15 = \mathbf{11.25}$

STEP 4 — C_0

$C_0 = 0.375(50) - 11.25/1.05 = 18.75 - 10.714$

ANSWER

$\mathbf{C_0 = 8.04}$ — vs. \$2.98 at lower vol. *Higher volatility raises the call value.*

Key observation

Option value depends on **volatility**, not on the real-world probabilities of up vs. down moves. The replicating portfolio (and therefore the no-arbitrage price) is independent of those probabilities.

LOS A · PUT VIA PARITY

Put Values via Put-Call Parity

Apply $S_0 + P_0 = C_0 + X/(1+r)^T$

At 20% volatility

$50 + P_0 = 2.98 + 55/1.05$

$P_0 = 2.98 + 52.38 - 50$

$\mathbf{P_0 \approx 5.36}$

At 40% volatility

$50 + P_0 = 8.04 + 55/1.05$

$P_0 = 8.04 + 52.38 - 50$

$\mathbf{P_0 \approx 10.42}$

Insight

As volatility doubles, both the call and the put roughly double — consistent with σ ↑ raising BOTH option premia (LM08 factor 5). The binomial model is internally consistent with parity.

QUIZ 2 · VOLATILITY & PARITY EXTENSIONS

Check Your Understanding — Quiz 2

Q6. With R^u=1.4, R^d=0.6 and the IFT setup, h equals:

Q7. With higher vol (R^u=1.4), C_0 equals:

Q8. Using parity with C_0=2.98 (20% vol), the put price P_0 is approximately:

Q9. Binomial option values depend on:

Q10. Doubling the width of the up/down moves (higher vol) causes the call value to:

LOS B · RISK-NEUTRAL PRICING

Risk-Neutral Probability π

Formula

$$\pi = (\mathbf{1} + \mathbf{r} - \mathbf{R^d}) / (\mathbf{R^u} - \mathbf{R^d})$$

Option value is the expected payoff under π, discounted at r:

$$\mathbf{C_0} = [\pi \mathbf{c^u} + (\mathbf{1} - \pi)\mathbf{c^d}] / (\mathbf{1} + \mathbf{r})^{\mathbf{T}}$$

Why "risk-neutral"?

π depends *only* on the up/down factors and r — NOT on investor risk preferences or real-world probabilities. In the risk-neutral world, every asset earns r. Real-world and risk-neutral probabilities rarely match, but the pricing result is the same as hedge-ratio replication.

Exam Tip: π is the probability that would make the expected return on the stock equal r (no risk premium). That's why it's called "risk-neutral."

LOS B · RISK-NEUTRAL WORKED EXAMPLES

Verification — Two Examples

WORKED EXAMPLE 4 **MM example (S_0=80, X=100, R^u=1.375, R^d=0.75, r=5%)**

STEP 1 — π

$\pi = (1.05 - 0.75)/(1.375 - 0.75) = 0.30/0.625 = \mathbf{0.48}$

STEP 2 — C_0

$C_0 = [0.48(10) + 0.52(0)] / 1.05 = 4.80/1.05$

ANSWER

C_0 = 4.57 — matches the hedge-ratio result

WORKED EXAMPLE 5 **IFT example (S_0=50, X=55, R^u=1.2, R^d=0.8, r=5%)**

STEP 1 — π

$\pi = (1.05 - 0.80)/(1.20 - 0.80) = 0.25/0.40 = \mathbf{0.625}$ · $1 - \pi = 0.375$

STEP 2 — C_0

$C_0 = [0.625(5) + 0.375(0)] / 1.05 = 3.125/1.05$

ANSWER

C_0 ≈ 2.98 — matches hedge-ratio result

QUIZ 3 · RISK-NEUTRAL PROBABILITY

Check Your Understanding — Quiz 3

Q11. Risk-neutral probability π is defined as:

Q12. In the MM example (R^u=1.375, R^d=0.75, r=5%), π equals:

Q13. In the IFT example (R^u=1.2, R^d=0.8, r=5%), π equals:

Q14. Applying π = 0.625 with c^u=5, c^d=0, r=5%, the call value is:

Q15. In risk-neutral pricing, π depends on:

LM10 · SCORECARD

Your Quiz Score

0 / 15

Key Takeaways

- **Hedge ratio**: $h = (c^u - c^d)/(S^u - S^d)$
- **Replication pricing**: $C_0 = hS_0 - V^u/(1+r)^T$
- **Risk-neutral probability**: $\pi = (1+r - R^d)/(R^u - R^d)$
- **Risk-neutral pricing**: $C_0 = [\pi c^u + (1-\pi)c^d]/(1+r)^T$

- Option value depends on *volatility, not real-world probabilities*.

Last-Minute Checklist

1. MM example: S_0=80, X=100, R^u=1.375, R^d=0.75, r=5% → h=0.2, V=12, C_0=4.57, π=0.48.
2. IFT example: S_0=50, X=55, R^u=1.2, R^d=0.8, r=5% → h=0.25, V=10, C_0=2.98, π=0.625.
3. Higher vol (R^u=1.4, R^d=0.6): h=0.375, C_0=8.04 — vol raises option value.
4. Puts via parity: P_0 = 5.36 (20% vol), P_0 = 10.42 (40% vol).
5. Both methods (hedge-ratio and risk-neutral) give the same C_0.

Step 1 of 13