LM01 · PORTFOLIO MANAGEMENT · CFA LEVEL I

Portfolio Risk and Return: Part I

Return measures, the risk-return tradeoff, correlation, efficient frontier, CAL and optimal portfolio selection

What This Reading Covers

All financial assets have **two common characteristics**: (1) an expected return, and (2) uncertainty regarding that return (risk). All financial assets can therefore be described by **risk & return**. This reading develops the tools to measure both and the framework for combining risky and risk-free assets into an optimal portfolio.

Learning Outcome Statements

- **LOS a** Describe characteristics of the major asset classes considered when forming portfolios
- **LOS b** Calculate and interpret the mean, variance, and covariance (or correlation) of asset returns based on historical data
- **LOS c** Explain risk aversion and its implications for portfolio selection
- **LOS d** Calculate and interpret portfolio standard deviation
- **LOS e** Describe the effect on a portfolio's risk of investing in assets that are less than perfectly correlated
- **LOS f** Describe and interpret the minimum-variance and efficient frontiers of risky assets and the global minimum-variance portfolio
- **LOS g** Explain the selection of an optimal portfolio, given an investor's utility (or risk aversion) and the capital allocation line

Exam Tip: The highest-frequency testables are the portfolio-variance formula (two-asset), the effect of $\rho<1$ on σ_P, the Markowitz efficient frontier and global minimum-variance portfolio, the CAL (with its slope = Sharpe ratio), and the utility function $U = E(R) - \frac{1}{2}A\sigma^2$.

LOS A · MAJOR ASSET CLASSES

Risk and Return of Major Asset Classes

Based on U.S. data over 1926–2017, small-capitalization stocks have had the **greatest average returns and greatest risk** over the period. T-bills had the **lowest average returns and the lowest standard deviation** of returns. Results for other markets around the world are similar: *asset classes with the greatest average returns also have the highest standard deviations of returns*.

ASSET CLASS	AVG ANNUAL RETURN (GEOM. MEAN)	STD DEV (ANNUALIZED MONTHLY)
Small-cap stocks	12.1%	31.7%
Large-cap stocks	10.2%	19.8%
Long-term corporate bonds	6.1%	8.3%
Long-term government bonds	5.5%	9.9%
Treasury bills	3.4%	3.1%

Takeaway

A positive return-risk relationship holds across asset classes historically. Equities (especially small caps) compensate investors with higher expected returns in exchange for higher volatility; T-bills sit at the other end of the spectrum.

LOS B · RETURN CALCULATIONS

Return Measures

Return is typically derived from two sources: **income** (dividend yield) and **capital gains/losses**.

Single-Period: Holding Period Return (HPR)

HOLDING PERIOD RETURN

$$\text{HPR} = \frac{P_1 - P_0 + \text{Income}}{P_0} = \frac{P_1}{P_0}(1 + \text{div yield}) - 1$$

P_1	ending price	P_0	beginning price	*Income*	dividend or interest received

Multi-Period Measures

ARITHMETIC MEAN

Simple average — tells us only the average return over a random 1-period time frame. **Assumes no compounding.**

$$\bar{R}_{\text{arith}} = \frac{R_1 + R_2 + \cdots + R_T}{T}$$

GEOMETRIC MEAN

Represents **growth over a given time period** — considers compounding.

$$\bar{R}_{\text{geom}} = [(1 + R_1)(1 + R_2)\cdots(1 + R_T)]^{1/T} - 1$$

WORKED EXAMPLE Arithmetic vs Geometric on $1

GIVEN

R_1 = −50%, R_2 = 35%, R_3 = 27%. Start $1 \rightarrow afterYr1$:0.50 → Yr2: $0.675 \rightarrow Yr3$:0.85725.

ARITHMETIC MEAN

(−0.50 + 0.35 + 0.27)/3 = **4%**

GEOMETRIC MEAN

$[(0.50)(1.35)(1.27)]^{1/3} - 1 = (0.85725)^{1/3} - 1 =$ **−5.00%**

TAKEAWAY

Arithmetic (+4%) wildly overstates what the investor actually earned (−5%). The geometric mean correctly reflects the compounded growth of $1.

LOS B · MONEY-WEIGHTED AND TIME-WEIGHTED RETURNS

Money-Weighted and Time-Weighted Returns

MONEY-WEIGHTED RETURN (IRR)

The IRR of the portfolio's cash flows — **accurately reflects what a specific investor earned**, but **lacks comparability** because results depend on timing/size of investor contributions.

$$\sum_{t=0}^{T} \frac{CF_t}{(1+\mathrm{IRR})^t} = 0$$

TIME-WEIGHTED RATE OF RETURN

Measures the **compounded rate of growth of $1** over the measurement period — the **basis for comparing manager performance** (strips out cash-flow timing).

$$(1+\mathrm{TWRR})^T = (1+\mathrm{HPR}_1)(1+\mathrm{HPR}_2)\cdots(1+\mathrm{HPR}_T)$$

WORKED EXAMPLE **Smaller funds disadvantaged by MWRR**

GIVEN — 5-YR AUM & RETURN

YEAR	AUM	RETURN
1	$30M	15%
2	$45M	−5%
3	$20M	10%
4	$25M	15%
5	$35M	3%

HOLDING PERIOD RETURN / ARITHMETIC / GEOMETRIC

HPR (cumulative) = **42.35%**

Arithmetic mean = **7.6%**

Geometric mean (TWRR) = **7.32%**

Money-weighted rate (IRR) = **5.86%** (using CFs: −30, −10.5, +22.75, −3, −6.25, +36.05)

TAKEAWAY

TWRR (7.32%) > MWRR (5.86%) because AUM was smallest when returns were highest. TWRR is the fair benchmark for comparing managers.

LOS B · GROSS/NET, PRE/AFTER-TAX, REAL, LEVERAGED

Other Return Measures

GROSS VS NET RETURNS

Gross return = total return − trading fees.

Net return = gross return − management/admin fees.

Gross is the basis for comparing managers; net is what the investor earns.

PRE-TAX VS AFTER-TAX NOMINAL

Components of the gain matter — capital gains (short vs long term) and income (interest vs dividends) are taxed differently. Long-term cap gains and qualified dividends typically receive preferential tax treatment.

Real Returns

$$(1 + \text{nominal}) = (1 + \text{real risk-free}) \times (1 + \text{inflation premium}) \times (1 + \text{risk premium})$$

$$(1 + \text{real after-tax}) = \frac{1 + \text{nominal after-tax}}{1 + \text{inflation}}$$

WORKED EXAMPLE **Real After-Tax Return**

GIVEN

Nominal pre-tax R = 20%, inflation = 2%, tax rate = 30% (implied). Calculate real after-tax return.

SOLUTION

After-tax nominal ≈ 20% × (1 − 0.30) = 14%. Then real = 1.14 / 1.02 − 1 = **11.76%**. (In MM source: specific parameters give ≈ 8.39% under different tax/inflation assumptions.)

Leveraged Returns

Achieved via derivatives or margin. **Gains are magnified, as are losses**, and you must account for margin loan interest. Leverage increases both expected return and volatility proportionally.

LOS B · MEASURES OF RISK

Variance, Covariance, Correlation

Variance & Standard Deviation

POPULATION

$$\sigma^2 = \Sigma(R - \mu)^2 / N$$

$$\sigma = \quad \sigma^2$$

SAMPLE

$$s^2 = \Sigma(R - \bar{R})^2 / (N - 1)$$

$$s = \quad s^2$$

Since we typically don't know population parameters, we use sample statistics.

Covariance & Correlation

$$\mathrm{Cov}(R_i, R_j) = \mathrm{E}[(R_i - \mathrm{E}(R_i)) \cdot (R_j - \mathrm{E}(R_j))]$$

$$\rho_{ij} = \mathrm{Cov}(R_i, R_j) / (\sigma_i \cdot \sigma_j) \quad \in [-1, +1]$$

Variance of a Portfolio of Assets

A portfolio's variance = **sum of all the variances + all the covariances**. Variances and covariances are additive.

$$\sigma_{\mathrm{P}}^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \mathrm{Cov}(R_1, R_2)$$

$$\sigma_{\mathrm{P}}^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \rho_{12} \sigma_1 \sigma_2$$

LOS D · PORTFOLIO STD DEVIATION — WORKED EXAMPLE

Two-Asset Portfolio Example

WORKED EXAMPLE

S&P 500 + MSCI World Portfolio

GIVEN

ASSET	WEIGHT	E(R)	Σ
S&P 500	80%	9.93%	16.21%
MSCI	20%	18.20%	33.11%

Cov(S&P, MSCI) = 0.005

PORTFOLIO EXPECTED RETURN

$E(R_P) = w_1R_1 + w_2R_2 = 0.8(0.0993) + 0.2(0.1820) =$ **0.1158 or 11.58%**

PORTFOLIO VARIANCE

$\sigma_P^2 = (0.8)^2(0.1621)^2 + (0.2)^2(0.3311)^2 + 2(0.8)(0.2)(0.005)$

$= 0.01682 + 0.00439 + 0.00160$

= **0.02281**

PORTFOLIO STD DEV

$\sigma_P = \sqrt{0.02281} =$ **15.1%** — less than the 80/20 weighted average (20.59%) because $\rho < 1$ creates the diversification benefit.

QUIZ 1 · CHECKPOINT

Test Your Understanding: Returns, Variance, Covariance

Q1. A portfolio's holding period returns over three years are −50%, +35%, +27%. The *geometric mean* return is closest to:

Q2. An investor earning a high TWRR of 7.32% but a lower MWRR of 5.86% over five years most likely:

Q3. For a two-asset portfolio with weights $w_1 = 0.8$, $w_2 = 0.2$, $\sigma_1 = 16.21\%$, $\sigma_2 = 33.11\%$, and Cov = 0.005, portfolio variance is closest to:

Q4. Based on U.S. data 1926–2017, which asset class had the *highest* geometric mean return *and* the highest standard deviation?

Q5. The correlation coefficient ρ_{ij} is bounded between:

LOS C · RISK AVERSION AND UTILITY

Risk Aversion, Risk Tolerance, Utility

The "Sure thing vs Gamble" Thought Experiment

Given a sure 25vs.a50/50gamblepaying0 or 50(bothE[] = $25):

- **Risk-averse** — takes the sure thing; max return for a given level of risk / min risk for a given return.
- **Risk-seeking** — takes the gamble; gets satisfaction from uncertainty.
- **Risk-neutral** — indifferent; seeks higher return regardless of risk.

Risk Tolerance

The level of risk *willingly accepted* to achieve investment goals. **Lower risk tolerance ⇒ lower level of acceptable risk ⇒ higher risk aversion.**

Utility Function (Expected Utility)

$$U = \mathrm{E}(R) - ½A\sigma^2$$

E(R) expected return

A measure of risk aversion — marginal return required to accept additional risk

σ^2 variance of returns

A > 0	A = 0	A < 0
Risk-averse	Risk-neutral	Risk-seeking ("ignorance")

Key Conclusions

- Utility is **unbounded** on both sides.
- Higher return ⇒ higher utility.
- Higher risk ⇒ lower utility.
- The higher the value of A, the greater the negative effect of σ^2 on U.

WORKED EXAMPLE **Utility across four investments & three A values**

GIVEN — FOUR INVESTMENTS

INV.	E(R)	Σ
1	12%	30%
2	15%	35%
3	21%	40%
4	24%	45%

UTILITY = E(R) − ½AΣ²

INV.	A = 4	A = 2	A = 0
1	−0.06	0.03	0.12
2	−0.095	0.0275	0.15
3	−0.11	0.05	0.21
4	−0.165	0.0375	0.24 ✓

TAKEAWAY

A highly risk-averse investor (A=4) picks Investment 1 (U = −0.06, the least negative). A moderately risk-averse investor (A=2) picks Inv. 3 (U = 0.05). A risk-neutral investor (A=0) always picks the highest E(R): Inv. 4.

LOS C, G · INDIFFERENCE CURVES

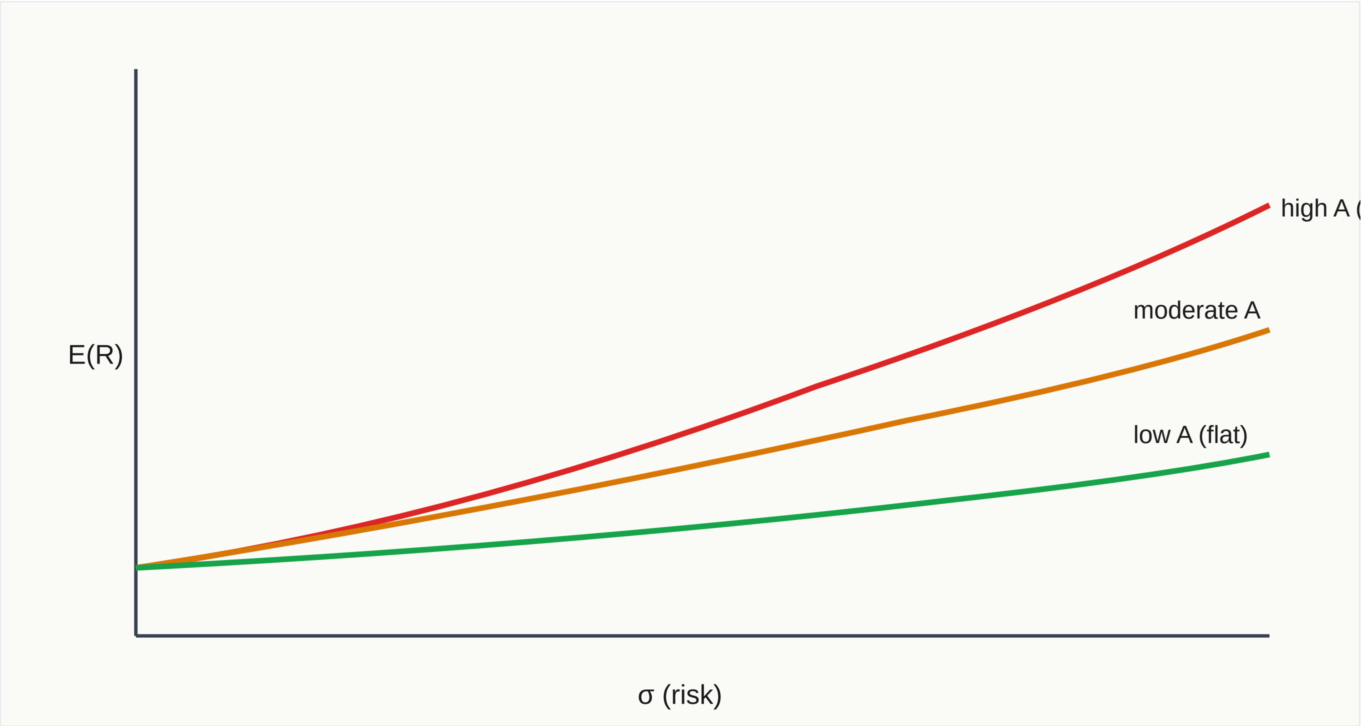
Indifference Curves

An indifference curve plots combinations of risk and expected return among which an investor is indifferent — they all have equal expected utility.

Shape

- **Upward sloping** — higher risk must be accompanied by higher return.
- **Curved (non-linear)** — increasing slope; required return increases at an increasing rate (*diminishing marginal utility of wealth*).
- Indifference curves **higher / to the left** ⇒ higher utility.

A **more risk-averse investor has steeper indifference curves** (e.g., A=4 steeper than A=2) — she requires a larger increment in expected return to bear additional volatility.



LOS E · CORRELATION & PORTFOLIO RISK

Effect of Correlation on Portfolio Risk

Correlation is the **key** driver of diversification. Letting $\rho_{12} = +1$, portfolio σ reduces to the weighted average of individual σ. For **any $\rho < +1$, $\sigma_P <$ weighted-avg σ**.

WORKED EXAMPLE **Two equal assets — ρ varies from +1 to −1**

GIVEN

$R_1 = R_2 = 10\%$, $\sigma_1 = \sigma_2 = 20\%$, $w_1 = w_2 = 50\%$. $R_P = 10\%$ in all cases.

P = +1

$\sigma_P^2 = (0.5)^2(0.2)^2 + (0.5)^2(0.2)^2 + 2(0.5)(0.5)(1)(0.2)(0.2) = 0.04$

$\sigma_P =$ **20%** (full weighted avg — no diversification)

P = 0

$\sigma_P^2 = (0.5)^2(0.2)^2 + (0.5)^2(0.2)^2 = 0.02$

$\sigma_P = \sqrt{0.02} =$ **14.1%**

P = −1

$\sigma_P^2 = 0.02 - 0.02 = 0$

$\sigma_P =$ **0%** — perfect hedge eliminates risk entirely!

TAKEAWAY

Same expected return (10%), dramatically different risk. The *less correlated* the assets, the *more risk-reduction* from combining them.

Practical Diversification

1. **Invest across asset classes** — stocks vs. bonds vs. cash vs. real assets.

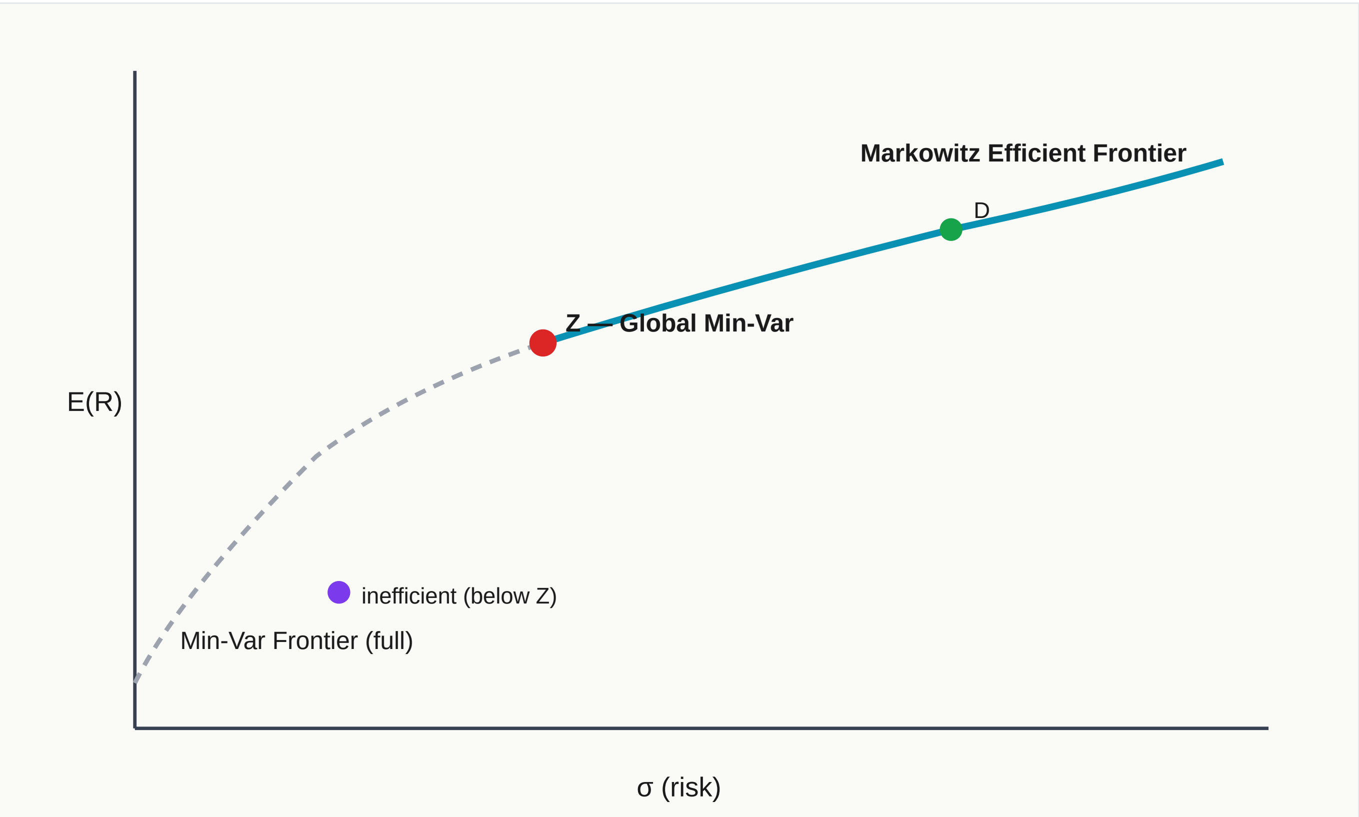
2. **Use index ETFs** — minimize costs of diversification.
3. **Diversify across countries** — different industries and business-cycle phases (energy vs pharma, corp vs gov't, large vs small cap).
4. **Don't own employer's stock** — pension plan should not own sponsor's stock as well.
5. **Add if risk-adjusted return benefits the portfolio.**
6. **Buy insurance** — e.g., buying put options.

Exam Tip: During financial crises, correlations tend to **increase** toward +1, which reduces the benefits of diversification precisely when investors most need them.

LOS F · MINIMUM-VARIANCE & EFFICIENT FRONTIERS

Efficient Frontier & Global Minimum-Variance Portfolio

The **Markowitz efficient frontier** represents the set of possible portfolios that have the **greatest expected return for each level of risk** (standard deviation of returns) — equivalently, the minimum risk for each level of return.



Key Vocabulary

- **Minimum-variance frontier** — all portfolios that provide the minimum variance for their given level of expected return (risky assets only).
- **Global Minimum-Variance Portfolio (Z)** — the portfolio on the min-var frontier with the absolute lowest σ. Everything *below* Z is dominated (inefficient).
- **Markowitz efficient frontier** — the *upper* portion of the min-var frontier, from Z to the top-right. These are all efficient because no portfolio offers higher return at the same (or lower) risk.

Finding the Global Minimum-Variance Weights (2 assets)

Set $d\sigma_P^2/dw_1 = 0$ and solve:

$$w_1* = [\sigma_2^2 - \mathrm{Cov}(R_1, R_2)]/[\sigma_1^2 + \sigma_2^2 - 2 \cdot \mathrm{Cov}(R_1, R_2)]$$

LOS G · CAL & TWO-FUND SEPARATION

Capital Allocation Line (CAL)

Begin with two assets: ① risk-free ($\sigma_f = 0$) and ② a risky portfolio (i). Let w_1 be the weight in the risk-free asset.

$$\mathrm{E}(R_\mathrm{P}) = w_1 \cdot R_f + (1 - w_1) \cdot \mathrm{E}(R_i)$$

$$\sigma_\mathrm{P}^2 = w_1^2 \cdot 0^2 + (1 - w_1)^2 \sigma_i^2 + 2w_1(1 - w_1) \cdot \mathrm{Cov}(R_f, R_i) = (1 - w_1)^2 \sigma_i^2$$

$$\sigma_\mathrm{P} = (1 - w_1) \cdot \sigma_i \quad \Rightarrow \quad (1 - w_1) = \sigma_\mathrm{P}/\sigma_i$$

Substituting and simplifying gives the **Capital Allocation Line**:

$$\mathrm{E}(R_\mathrm{P}) = R_f + [(\mathrm{E}(R_i) - R_f)/\sigma_i] \cdot \sigma_\mathrm{P}$$

The slope of the CAL = **$(E(R_i) - R_f) / \sigma_i$ = Sharpe ratio** of the risky portfolio. This is also called the **"market price of risk"** — the extra return per unit of σ an investor earns by taking risk.

Two-Fund Separation Theorem

All investors, regardless of taste, risk preferences, or wealth, **will hold a combination of two portfolios**: a risk-free asset and a risky "tangency" portfolio P.

The investment decision (identify the optimal risky portfolio) is separable from **the financing decision** (how much to lend at R_f or borrow at R_f to leverage P). Every point on an inferior CAL(A) is dominated by a point on the optimal CAL(P) with higher $E(R_P)$ for the same σ.

LOS A–G · PUTTING IT ALL TOGETHER

Comprehensive Worked Problem

COMPREHENSIVE **Two Risky Assets, ρ = 0, add R_f = 3%**

GIVEN

ASSET	$E(R_I)$	Σ
A	20%	50%
B	15%	33%

Correlation $\rho_{AB} = 0$.

STEP 1 — PORTFOLIO AT W_A = 10%

$E(R_P) = 0.1(0.20) + 0.9(0.15) =$ **0.155 or 15.5%**

$\sigma_P = \sqrt{[(0.1)^2(0.5)^2 + (0.9)^2(0.33)^2]} = \sqrt{(0.0025 + 0.08821)} =$ **30.12%**

STEP 2 — GLOBAL MINIMUM-VARIANCE

$w_A^* = [\sigma_B^2 - \text{Cov}] / [\sigma_A^2 + \sigma_B^2 - 2\cdot\text{Cov}] = (0.33^2 - 0) / (0.50^2 + 0.33^2 - 0) = 0.1089 / 0.3589 =$ **w_A^* = 0.30**, $w_B^* = 0.70$

STEP 3 — ADD R_F = 3%. CONSTRUCT CAL

Example tangency portfolio (from MM notes): optimal risky portfolio has σ ≈ 12.1% with E(R) ≈ 9% giving Sharpe = (0.09 − 0.03) / 0.121 ≈ 0.496.

CAL: $E(R_P) = 0.03 + 0.496 \cdot \sigma_P$.

STEP 4 — WHAT Σ IS NEEDED FOR E(R) = 20%?

$0.20 = 0.03 + 0.4978 \cdot \sigma_P \Rightarrow \sigma_P = (0.20 - 0.03) / 0.4978 =$ **34.2%**

Compared to **50%** (holding Asset A alone) — a huge risk reduction for the same expected return by levering into the optimal risky portfolio instead.

UTILITY CHECK (A = 2.5) ALONG THE CAL

Σ	E(R)	U = E(R) − ½(2.5)Σ²
0%	3%	0.030
12.1%	9%	0.0717
24.1%	15%	**0.0774 ✓ (max)**
34.2%	20%	0.0546

The optimal point for this investor is σ ≈ 24.1%, E(R) ≈ 15%.

QUIZ 2 · CHECKPOINT

Test Your Understanding: Utility, Efficient Frontier, CAL

Q6. Which of the following statements about an investor's utility function $U = E(R) - \frac{1}{2}A\sigma^2$ is *most accurate*?

Q7. The *Global Minimum-Variance Portfolio* is *best* described as:

Q8. The Capital Allocation Line (CAL) is a straight line because:

Q9. The slope of the CAL equals:

Q10. The Two-Fund Separation Theorem says that:

LOS F, G · SHARPE RATIO AND DOMINANCE

Sharpe Ratio, Dominance, and the Optimal CAL

Sharpe Ratio

Also called the "reward-to-variability ratio":

$$Sharpe = [\mathrm{E}(R_i) - R_f]/\sigma_i$$

Higher Sharpe ⇒ better risk-adjusted return. When we add a risk-free asset, the **optimal CAL** is the one with the highest Sharpe ratio — the line from R_f that is tangent to the efficient frontier at portfolio P.

Dominance Logic

Among risky portfolios on the efficient frontier, those along CAL(P) **dominate** points on any inferior CAL(A):

- For every σ on CAL(A), there is a point on CAL(P) with **higher E(R)** for the same σ.
- Equivalently, for every E(R), CAL(P) offers **lower σ** than CAL(A).

Leverage Above P

Points on the CAL **above** tangency portfolio P are achieved by **leveraging P** — borrowing at R_f and investing additional funds in P. Points **below** P are achieved by **lending** at R_f (holding both P and the risk-free asset).

Exam Tip: On the exam, when asked to compare two portfolios both on the efficient frontier, the one with the **higher Sharpe ratio wins** if combined with a risk-free asset — regardless of absolute E(R) or σ alone.

Why n stocks don't scale infinitely

Finding global min-var weights becomes unwieldy because a portfolio of n = 50 assets requires 50 variances **plus 1,225 covariances** [n(n−1)/2]. Rather than computing pairwise σ_{ij} among all stocks, analysts compare each stock with a **benchmark** — needing only n betas (50 instead of 1,275 parameters).

QUIZ 3 · CHECKPOINT

Test Your Mastery: Comprehensive

Q11. Two assets are 50/50 weighted with $\sigma_1 = \sigma_2 = 20\%$. Portfolio standard deviation when $\rho_{12} = 0$ is *closest to*:

Q12. Which portfolio on the minimum-variance frontier is inefficient?

Q13. An investor with utility function $U = E(R) - \frac{1}{2}A\sigma^2$, where $A = 2.5$, evaluates four points on a CAL ($R_f = 3\%$, slope = 0.4978). She chooses:

Q14. Portfolio X has Sharpe = 0.40; Portfolio Y has Sharpe = 0.60. A risk-averse investor who can also invest in a risk-free asset will prefer to allocate along the CAL drawn through:

Q15. During a financial crisis, portfolio diversification benefits typically:

MODULE COMPLETE

Your Scorecard

0 / 15

LM01 Key Takeaways — Exam-Ready Cheat Sheet

Return Measures

1. **HPR**: $(P_1 - P_0 + \text{Inc})/P_0$ · **Arithmetic**: simple avg · **Geometric**: $[(1+R_1)\ldots(1+R_T)]^{1/T} - 1$
2. **MWRR (IRR)** reflects *this* investor's experience. **TWRR** compounds period returns — use to compare managers.
3. Small-cap > Large-cap > LT Corp > LT Gov > T-bills on both return and risk (1926–2017).

Risk, Variance & Correlation

4. $\sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2$. $\rho \in [-1, +1]$.
5. For any $\rho < +1$, $\sigma_P <$ weighted-avg σ. At $\rho = -1$, risk can be driven to 0.

Utility & Optimal Portfolio

6. **$U = E(R) - \frac{1}{2}A\sigma^2$**. $A > 0$ risk-averse, $A = 0$ risk-neutral, $A < 0$ risk-seeking.
7. Indifference curves upward-sloping & curved; **steeper = more risk-averse**.
8. **Global Min-Var Portfolio** = leftmost point; **Markowitz efficient frontier** = upper arc only.
9. **CAL**: $E(R_P) = R_f + [(E(R_i) - R_f)/\sigma_i]\cdot\sigma_P$. Slope = Sharpe ratio = "market price of risk".
10. **Two-Fund Separation**: every investor holds a combination of R_f + the tangency portfolio P. The optimal CAL has the highest Sharpe.

Last-Minute Exam Checklist

TWRR vs MWRR: TWRR compares managers; MWRR reflects investor's cash-flow timing

Portfolio variance: variances + covariances are additive

Correlation effect: $\rho=+1$ (no diversification), $\rho=0$ (some), $\rho=-1$ (perfect hedge, $\sigma=0$)

Efficient frontier: upper arc of min-var frontier; below global min-var = inefficient

CAL slope = Sharpe; optimal CAL = steepest Sharpe

Utility: $U = E(R) - \frac{1}{2}A\sigma^2$; higher A ⇒ steeper indifference curves

---

Step 1 of 18