LM02 · QUANTITATIVE METHODS · CFA LEVEL I

# Time Value of Money in Finance

PV of fixed-income and equity, implied returns and growth, cash flow additivity

### Three Rules of Money

1. **Larger cash flows are worth more**
2. **Less risky CFs are worth more** (lower discount rate)
3. **CFs sooner are worth more** (time value of money)

## Learning Outcome Statements

**LOS a** Calculate and interpret the PV of fixed-income and equity instruments based on expected future cash flows

**LOS b** Calculate and interpret the implied return of fixed-income instruments and required return and implied growth of equity instruments given the PV and cash flows

**LOS c** Explain the cash flow additivity principle, its importance for the no-arbitrage condition, and its use in calculating implied forward interest rates, forward exchange rates, and option values

**Exam Tip:** Every formula in this reading is a rearrangement of $PV = FV / (1 + r)^t$. Master the three equity models (constant-dividend perpetuity, Gordon growth, two-stage), the ZCB price/yield mechanics, and the implied forward rate relation $(1 + S_2)^2 = (1 + S_1)(1 + f_{1,1})$.

LOS A · PRESENT VALUE FOUNDATIONS

# PV, FV, and Continuous Compounding

TIME VALUE OF MONEY

$$FV = PV(1+r)^t \qquad PV = \frac{FV}{(1+r)^t} = FV(1+r)^{-t}$$

| | | | | | |
|---|---|---|---|---|---|
| $PV$ | present value | $FV$ | future value | $r$ | per-period discount rate |
| $t$ | number of periods | | | | |

$$\text{Continuous:} \quad FV = PV \cdot e^{rT} \qquad PV = FV \cdot e^{-rT}$$

## Fixed-Income / Debt Instrument Types

| TYPE | CASH-FLOW PATTERN |
|---|---|
| **Zero-coupon bond (ZCB)** | T-Bills (≤ 1 yr); sold at discount, mature at par — *single CF at maturity* |
| **Coupon bond (Notes, Bonds)** | Interest payments over time + par at maturity |
| **Fully amortizing** | Mortgage, auto loan — level payments of interest and principal |

## Equity Instrument Types

**Preferred shares** Constant dividend → perpetuity valuation

**Mature common** Growing dividend (constant rate) → Gordon growth model

**Growth stocks** Non-constant growth → two-stage DDM

LOS A · ZERO-COUPON BONDS

# Zero-Coupon Bond Valuation

$$PV_{ZCB} = \frac{FV}{(1+r)^T}$$

where r = discount rate, IRR, or YTM.

WORKED EXAMPLE **20-Year ZCB at 6.7% YTM**

**GIVEN**

20-year ZCB with par = 100 and YTM = 6.7%.

**SOLUTION**

PV = 100 / $(1.067)^{20}$ = **$27.33**

Calculator: FV=100, I/Y=6.7, PMT=0, N=20 → CPT PV

WORKED EXAMPLE **Price in 3 Years, Same YTM**

**GIVEN**

Same 20-yr ZCB; YTM held at 6.7%; find $P_3$.

**SOLUTION (TWO EQUIVALENT PATHS)**

(a) Grow PV forward: 27.33 × $(1.067)^3$ = **$33.21**

(b) Discount par over 17 remaining years: 100 / $(1.067)^{17}$ = **$33.21**

WORKED EXAMPLE **Implied YTM from Observed Price**

**GIVEN**

20-yr ZCB, par = 100, PV = \$22.68224.

**SOLUTION**

YTM = $(100/22.68224)^{1/20} - 1$ = **7.70%**

**Exam Tip:** Negative yields exist — a 10-yr ZCB at −0.05% gives PV = $100/(0.9995)^{10} \approx 100.50$.

CHECK YOUR UNDERSTANDING · QUIZ 1

# Quiz 1

Select the best answer. Feedback appears after each click.

**Q1. A 20-year zero-coupon bond with $100 par trades at a YTM of 6.7%. What is its price?**

**Q2. A coupon bond with coupon rate < YTM trades at:**

**Q3. $D_0$ = 1.50, g = 6% constant forever, r = 15%. Gordon growth value =**

**Q4. For a perpetuity with PMT = $0.825 per quarter and P =$97.03, annualized r =**

**Q5. A $800K 30-year mortgage at 5.25% annual rate has monthly payment approximately:**

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LOS A · COUPON BONDS & ANNUITIES

## Coupon Bonds, Perpetuities, and Amortizing Annuities

### Coupon Bond Price

$$\mathrm{PV} = \frac{\mathrm{PMT}_1}{1+r} + \frac{\mathrm{PMT}_2}{(1+r)^2} + \cdots + \frac{\mathrm{FV} + \mathrm{PMT}_N}{(1+r)^N}$$

Par bond: c = r ⇒ price = par. Premium: c > r. Discount: c < r.

WORKED EXAMPLE **20-Yr 6.7% Semi-Annual Bond at 7.7% YTM**

GIVEN

Par = 100, coupon = 6.70% p.a. (semi-annual), YTM = 7.70%, N = 20×2 = 40.

**SOLUTION**

PMT = 3.35, I/Y (per period) = 3.85, N = 40 → PV = **89.88** (trades at a discount because coupon < YTM).

### Perpetuity (preferred shares, some bonds)

$$\mathrm{PV} = \frac{\mathrm{PMT}}{r}$$

**WORKED EXAMPLE** **Quarterly-Coupon Perpetuity**

**GIVEN**

3.3% annual coupon paid quarterly (0.825 per quarter) and P = 97.03.

**SOLUTION**

$r_4$ = 0.825 / 97.03 = 0.8502% per quarter → annualized r = **3.401%**

### Amortizing Annuity (mortgage / car loan)

ANNUITY PAYMENT

$$A = \frac{r \cdot \mathrm{PV}}{1 - (1 + r)^{-t}}$$

| | | | | | |
|---|---|---|---|---|---|
| $A$ | periodic level payment | $PV$ | loan principal (present value) | $r$ | per-period interest rate |
| $t$ | total number of payments | | | | |

**WORKED EXAMPLE** **$800K, 30-Yr FRM at 5.25%**

**GIVEN**

PV = −800,000, N = 30×12 = 360, I/Y = 5.25/12 = 0.4375%, FV = 0.

**SOLUTION**

CPT PMT = **$4,417.63 / month**

Amortization month 1: interest = 800,000 × 0.4375% = 3,500.00; principal = 917.63; balance = 799,082.37.

Each subsequent month the interest component falls and the principal component rises.

LOS A · EQUITY VALUATION (DDM)

# Dividend Discount Models for Equity

## 1. Constant Dividend (Perpetuity)

$$\text{PV} = \frac{D}{r}$$

Many REITs and preferred shares. e.g., D = 1.50, r = 15% → PV = 1.50/0.15 = **$10**.

## 2. Gordon Growth Model (constant-growth dividend)

GORDON GROWTH MODEL

$$V_0 = \frac{D_0(1+g)}{r-g} = \frac{D_1}{r-g}$$

| | | | | | |
|---|---|---|---|---|---|
| V_0 | current stock value | D_0 | most recent dividend paid | D_1 | next expected dividend |
| g | constant dividend growth rate | r | required rate of return | | |

Requires g < r. For commercial real estate: Property value = $NOI_1$ / (r − g) where r − g = cap rate.

**WORKED EXAMPLE** Gordon Growth Valuation

**GIVEN**

$D_0$ = 1.50, g = 6%, r = 15%.

**SOLUTION**

PV = 1.50(1.06) / (0.15 − 0.06) = 1.59 / 0.09 = **$17.67**

## 3. Two-Stage DDM (non-constant growth)

$$V_0 = \sum_{i=1}^{n} \frac{D_0(1+g_s)^i}{(1+r)^i} + \frac{1}{(1+r)^n} \cdot \frac{D_0(1+g_s)^n(1+g_L)}{r-g_L}$$

**WORKED EXAMPLE** Two-Stage DDM

**GIVEN**

$D_0$ = 1.50, $g_s$ = 6% for 3 yrs, $g_L$ = 2% thereafter, r = 15%.

**SOLUTION**

$D_1$=1.59, $D_2$=1.6854, $D_3$=1.7865

Terminal at t=3: 1.7865 × 1.02 / (0.15 − 0.02) = 14.017

PV = $1.59/1.15 + 1.6854/1.15^2 + 1.7865/1.15^3 + 14.017/1.15^3$

= 1.3826 + 1.2744 + 1.1747 + 9.2166 = **$13.05**

LOS B · IMPLIED RETURN AND IMPLIED GROWTH

# Solving Backwards for r or g

## ZCB Implied Return

$$r = \left(\frac{\mathrm{FV}}{\mathrm{PV}}\right)^{1/T} - 1$$

WORKED EXAMPLE **ZCB Implied Returns**

**GIVEN**

At t=0 PV = 100.50; at t=6 the bond trades at 95.72. Par = 100, mat. = 10.

**SOLUTION**

Realized return over 6 yrs: $(95.72/100.50)^{1/6} - 1 =$ **−0.81%** p.a.

Current YTM (4 yrs remain): $(100/95.72)^{1/4} - 1 =$ **1.10%**

## Coupon Bond Implied Return (2-period example)

WORKED EXAMPLE **2-Yr Coupon Bond Realized Return**

**GIVEN**

2% annual coupon bond purchased at 100 with PMT = 2 reinvested at 2%; sold after 2 years at 93.091.

**SOLUTION**

Accumulated future value at t=2 = 2(1.02) + 2 + 93.091 = 97.131.

$r = (97.131/100)^{1/2} - 1 =$ **−1.45%**

## Equity Implied Required Return and Implied Growth (GGM rearrangements)

$$r = \frac{D_0(1+g)}{\text{PV}} + g \qquad g = r - \frac{D_0(1+g)}{\text{PV}}$$

**WORKED EXAMPLE** Implied Required Return on Equity

**GIVEN**

$P_0$ = 63, $D_1$ = 1.76, g = 4%.

**SOLUTION**

r = 1.76/63 + 4% = 2.79% + 4.00% = **6.79%**

### Forward P/E Relation

$$\frac{P_0}{E_1} = \frac{D_1/E_1}{r-g}$$

Dividend payout ratio plus (r, g) determine the forward multiple. Higher expected growth ⇒ higher multiple. Higher required return ⇒ lower multiple.

**WORKED EXAMPLE** Forward P/E Solves for r or g

**GIVEN**

(1) Forward P/E = 28, DPR = 70%, g = 4% → find r.
(2) Forward P/E = 19, DPR = 60%, r = 8% → find g.
(3) Forward P/E = 28, DPR = 70%, r = 9%, g = 4.5% → buy or sell?

**SOLUTION**

(1) $28 = 0.70 / (r - 0.04) \rightarrow r =$ **6.5%**

(2) $19 = 0.60 / (0.08 - g) \rightarrow g =$ **4.84%**

(3) Fair P/E = $0.70 / (0.09 - 0.045) =$ **15.5×**. Stock trades at 28× ⇒ **SELL**.

CHECK YOUR UNDERSTANDING · QUIZ 2

## Quiz 2

Select the best answer. Feedback appears after each click.

**Q1. A ZCB was priced at 100.50 six years ago and now trades at 95.72. Realized annual return =**

**Q2. Two-stage DDM: $D_0$=1.50, $g_S$=6% for 3 yrs, $g_L$=2%, r=15%. Value =**

**Q3. Forward P/E = 28, DPR = 70%, g = 4%. Implied required return r =**

**Q4. Two CF streams with equal PVs indicates:**

**Q5. A risk-free return implied between spot rates $S_1$=2.5% and $S_2$=3.5% is**

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LOS C · CASH FLOW ADDITIVITY & NO-ARBITRAGE

## The Cash Flow Additivity Principle

**Principle**

Two economically equivalent strategies must have the same price. Value CFs additively — take the difference (A − B) and price that difference. If PV(A − B) > 0, choose A; if < 0, choose B.

WORKED EXAMPLE **Choosing Between Two Cash-Flow Streams**

GIVEN

r = 6%. Strategy A: 45, 45, 45. Strategy B: 60, 40, 32.50. Find which has higher PV.

SOLUTION

$(A - B) = -15, 5, 12.50$. $PV(A - B) = -15/1.06 + 5/1.06^2 + 12.50/1.06^3 \approx 0$. ⇒ Both strategies are economically equivalent.

WORKED EXAMPLE

## Clearly Dominant Strategy

**GIVEN**

r = 6%. Strategy A: 100. Strategy B: 120 and −20 in year 1 (B is clearly superior).

**SOLUTION**

PV(A) = 100/1.06 = 94.34. PV(B) = 120/1.06 − 20/1.06 = 113.21 − 18.87 = 94.34 + 18.87 = **113.21**. Difference 18.87 favors B.

LOS C · IMPLIED FORWARD RATES

# Implied Forward Rates from the Spot Curve

### No-Arbitrage Spot/Forward Relation

IMPLIED FORWARD RATE

$$(1 + S_2)^2 = (1 + S_1)(1 + f_{1,1})$$

| | | | |
|---|---|---|---|
| $S_1$ | 1-year spot rate | $S_2$ | 2-year spot rate |
| $f_{1,1}$ | 1-year rate starting 1 year from now | | |

where $S_1$, $S_2$ are 1-yr and 2-yr spot rates, $f_{1,1}$ is the 1-yr rate starting 1 yr from now.

## WORKED EXAMPLE Deriving $f_{1,1}$

**GIVEN**

$S_1 = 2.5\%$, $S_2 = 3.5\%$. Find $f_{1,1}$.

**SOLUTION**

$f_{1,1} = (1.035)^2 / 1.025 - 1 =$ **4.51%**

**ARBITRAGE INTUITION**

If an FRA locks in 5% instead of 4.51%, arbitrage profit of $(1.025)(1.05) - (1.035)^2 = 0.005025$ per 100 par.

## WORKED EXAMPLE Change in $f_{1,1}$ Over Time

**GIVEN**

1-yr ZCB and 2-yr ZCB prices:

May 31: 98.028 / 95.109

June 15: 97.402 / 93.937

**SOLUTION**

May 31 spot: $S_1$ = (100/98.028) − 1 = 2.011%; $S_2 = (100/95.109)^{1/2} - 1$ = 2.539%.

May 31 $f_{1,1} = (1.02539)^2/(1.02011) - 1$ = 3.069%

June 15 spot: $S_1$ = 2.667%; $S_2$ = 3.1767%.

June 15 $f_{1,1} = (1.031767)^2/(1.026673) - 1$ = 3.689%

$\Delta f_{1,1}$ = 3.689% − 3.069% = **+61.96 bps**

LOS C · FORWARD EXCHANGE RATES

# Covered Interest Rate Parity — FX Forwards

### Continuous-Compounding Formula

$$F_{f/d} = S_{f/d} \cdot e^{r_f T} / e^{r_d T}$$

$$Simple: F_{f/d} = S_{f/d} \cdot (1 + r_f \cdot days/360)/(1 + r_d \cdot days/360)$$

Low-rate currency trades at a forward *premium*.

**WORKED EXAMPLE** JPY/USD 6-Month Forward

**GIVEN**

$S_{JPY/USD}$ = 134.40, $r_{USD}$ = 2%, $r_{JPY}$ = 0.05%, T = 6 months.

**SOLUTION**

$F_{JPY/USD} = 134.40 \cdot e^{0.0005(0.5)}/e^{0.02(0.5)}$ = **133.0959**

JPY trades at a forward premium vs. USD.

**WORKED EXAMPLE** 1-Yr Forward Over Two Dates

**GIVEN**

Spot = 1.2602. May 31: $r_f$ = 2.012%, $r_d$ = 1.291%. June 15: $r_f$ = 2.667%, $r_d$ = 1.562%.

**SOLUTION**

May 31 Fwd = $1.2602 \cdot e^{0.02012}/e^{0.01291}$ = 1.269319

June 15 Fwd = $1.2602 \cdot e^{0.02667}/e^{0.01562}$ = 1.274202

$\Delta F = +0.004883 =$ **+48.83 pips**

LOS C · ONE-PERIOD BINOMIAL OPTION PRICING

# Replicating Portfolio for Options

### Hedge Ratio and Call Value

$$h = (c^{+} - c^{-})/(S^{+} - S^{-})$$

$$c_0 = h \cdot S_0 - [h \cdot S^{+} - c^{+}]/(1 + r)$$

The hedge ratio h is the number of shares short to construct a risk-free portfolio with one long call.

WORKED EXAMPLE **One-Period Call Option**

GIVEN

$S_0$ = 40, u = 1.4, d = 0.8 → $S^{+}$ = 56, $S^{-}$ = 32. Call with X = 50: $c^{+}$ = 6, $c^{-}$ = 0. r = 5%.

SOLUTION

h = (6 − 0)/(56 − 32) = 6/24 = **0.25**

Risk-free payoff: $hS^{+} - c^{+}$ = 0.25(56) − 6 = 8 = $hS^{-} - c^{-}$ = 0.25(32) − 0 = 8 ✓

$c_0$ = 0.25(40) − 8/1.05 = 10 − 7.6190 = **$2.38**

WORKED EXAMPLE **One-Period Put Option**

GIVEN

Same tree; Put with X = 50: $p^{+}$ = 0 (at 56), $p^{-}$ = 18 (at 32).

SOLUTION

$h_{put}$ = (0 − 18)/(56 − 32) = **−0.75** (i.e. long 0.75 shares per long put)

0.75(40) + $p_0$ = 0.75(32) + 18 / 1.05

$p_0$ = (24 + 18)/1.05 − 30 = 42/1.05 − 30 = **$10**

**Exam Tip:** The no-arbitrage call price in one-period binomial is equivalent to the *risk-neutral* expectation: $c_0 = [\pi c^+ + (1 - \pi)c^-] / (1 + r)$, with $\pi = [(1 + r) - d] / (u - d)$.

CHECK YOUR UNDERSTANDING · QUIZ 3

## Quiz 3

Select the best answer. Feedback appears after each click.

**Q1. Spot JPY/USD = 134.40, $r_{USD}$ = 2%, $r_{JPY}$ = 0.05%, T = 0.5. $F_{JPY/USD}$ is closest to:**

**Q2. $S_0$ = 40; up factor 1.4 ($S^+$=56, $c^+$=6), down factor 0.8 ($S^-$=32, $c^-$=0), r = 5%. Hedge ratio h =**

**Q3. For the same tree, the one-period no-arbitrage call value $c_0$ is closest to:**

**Q4. Arbitrageur observes $S_1$=2.5%, $S_2$=3.5% and an FRA quote of $f_{1,1}$=5.0%. Best action:**

**Q5. Which statement on cash flow additivity is MOST accurate?**

MODULE COMPLETE

# LM02 · Time Value of Money in Finance Summary

### Final Score

**0 / 0**

### Cheat Sheet

$$\mathrm{FV} = \mathrm{PV}(1+r)^t; \mathrm{PV} = \mathrm{FV}/(1+r)^t; continuous: \mathrm{FV} = \mathrm{PV} \cdot e^{rT}$$

$$\mathrm{ZCB}:\mathrm{PV}=\mathrm{FV}/(1+r)^{T}; r=(\mathrm{FV}/\mathrm{PV})^{1/T}-1$$

$$Perpetuity:\mathrm{PV}=\mathrm{PMT}/r; Amortizing: A=r\cdot\mathrm{PV}/[1-(1+r)^{-t}]$$

$$Gordon:\mathrm{PV}=D_0(1+g)/(r-g); r=D_0(1+g)/\mathrm{PV}+g$$

$$Forwardrate:(1+S_2)^2=(1+S_1)(1+f_{1,1})$$

$$FX: F_{f/d}=S_{f/d}\cdot e^{(r_f-r_d)T}$$

$$Binomial: h=(c^{+}-c^{-})/(S^{+}-S^{-}); c_0=h\cdot S_0-(hS^{+}-c^{+})/(1+r)$$

### Last-Minute Exam Checklist

**ZCB:** 20-yr @ 6.7% → $27.33 · Implied r = $(FV/PV)^{1/T}-1$

**Gordon:** $D_0$=1.50, g=6%, r=15% → **$17.67**

**Two-stage DDM:** same inputs + $g_L$=2% after 3 yrs → **$13.05**

**Forward:** $f_{1,1}=(1.035)^2/1.025-1$ = **4.51%**

**FX:** JPY/USD 6-mo → **133.0959**

**Binomial call:** h=0.25, $c_0$=**$2.38**

Step 1 of 14