LM04 · QUANTITATIVE METHODS · CFA LEVEL I

Probability Trees and Conditional Expectations

Expected values, conditional expectations, Bayes' formula

Key Point

Investment decisions are made **under uncertainty**. The expected value E(X) is a probability-weighted average of outcomes — a forecast of future value and our best estimate of the true population mean.

Learning Outcome Statements

LOS a Calculate expected values, variances, and standard deviations and demonstrate their application to investment problems

LOS b Formulate an investment problem as a probability tree and explain the use of conditional expectations in investment application

LOS c Calculate and interpret an updated probability in an investment setting using Bayes' formula

Exam Tip: Bayes' formula and total probability rule carry heavy exam weight. Know: $P(A|B) = P(B|A)\cdot P(A)/P(B)$.

LOS A · EXPECTED VALUE

E(X), Variance, and Standard Deviation of a Random Variable

$$\mathrm{E}(X) = \Sigma_{i=1..n} \mathrm{P}(X_i) X_i$$

$$\sigma^2(X) = \Sigma\ \mathrm{P}(X_i)[X_i - \mathrm{E}(X)]^2 \quad \cdot \quad \sigma(X) = \quad \sigma^2(X)$$

Unlike sample variance which divides by n−1, the variance of a random variable uses **probability weights** — no degrees-of-freedom correction because we're describing the *entire* distribution.

WORKED EXAMPLE

EPS Probability Distribution

GIVEN

P = 0.15, EPS = 2.60 · P = 0.45, EPS = 2.45
P = 0.24, EPS = 2.20 · P = 0.16, EPS = 2.00

SOLUTION

E(EPS) = 0.15(2.60) + 0.45(2.45) + 0.24(2.20) + 0.16(2.00) = **2.3405**

σ^2 = 0.15(2.60−2.3405)^2 + 0.45(2.45−2.3405)^2 + 0.24(2.20−2.3405)^2 + 0.16(2.00−2.3405)^2 = **0.038785**

σ = √0.038785 = **0.196939**

WORKED EXAMPLE

Project Cash Flow (BAII Plus)

GIVEN

Good (P=0.3, CF=50); Average (P=0.5, CF=40); Weak (P=0.2, CF=20).

SOLUTION

$E(CF) = 0.3(50) + 0.5(40) + 0.2(20) = 15 + 20 + 4 = \mathbf{39}$

Via BAII [2nd][DATA] with X/Y pairs and 1-V mode: N=100, $\bar{X}$=**39**, σx=**10.44** ⇒ Variance ≈ 10.44^2 = **109.00**.

LOS B · PROBABILITY TREES

Probability Trees and Conditional Expectation

Total Probability Rule for Expectation

$$\mathrm{E}(X) = \mathrm{P}(S) \cdot \mathrm{E}(X|S) + \mathrm{P}(S^c) \cdot \mathrm{E}(X|S^c)$$

With mutually exclusive and exhaustive scenarios S, S^c, unconditional expected value is the probability-weighted combination of conditional expectations.

WORKED EXAMPLE **Rate-Dependent EPS Tree**

GIVEN

P(declining rates) = 0.60; P(stable rates) = 0.40.
If declining: P(2.60)=0.25, P(2.45)=0.75.
If stable: P(2.20)=0.60, P(2.00)=0.40.

SOLUTION

Joint probabilities: 0.25×0.60=**0.15**; 0.75×0.60=**0.45**; 0.60×0.40=**0.24**; 0.40×0.40=**0.16**. Total = 1.00.
E(EPS) = (0.25)(0.60)(2.60) + (0.75)(0.60)(2.45) + (0.60)(0.40)(2.20) + (0.40)(0.40)(2.00) = **2.3405**

WORKED EXAMPLE **Stock Price vs Interest Rate Path**

GIVEN

P(decline) = 0.4; if decline, P$(100) = 0.75, P($90)=0.25. If no decline, P$(80) = P($70)=0.50.

SOLUTION

E(Price | decline) = 0.75(100) + 0.25(90) = $97.50

E(Price | no decline) = 0.50(80) + 0.50(70) = $75.00

E(Price) = 0.40(97.50) + 0.60(75.00) = 39.00+45.00 = **$84.00**

LOS C · BAYES' FORMULA

Updating Prior Probabilities with New Information

Bayes' Formula

$$\mathrm{P}(Event|Information) = \mathrm{P}(Information|Event)/\mathrm{P}(Information) \cdot \mathrm{P}(Event)$$

Equivalently: $P(S_n \mid E) = P(E \mid S_n) \cdot P(S_n) / P(E)$. Updates *prior* to *posterior* given new evidence.

Total Probability Rule (denominator P(E))

$$\mathrm{P}(\mathrm{E}) = \mathrm{P}(\mathrm{E}|S_1)\mathrm{P}(S_1) + \mathrm{P}(\mathrm{E}|S_2)\mathrm{P}(S_2) + \ldots + \mathrm{P}(\mathrm{E}|S_n)\mathrm{P}(S_n)$$

WORKED EXAMPLE **P(Expansion | Stock Price Exceeds)**

GIVEN

Priors: P(Exp)=0.45, P(Moderate)=0.30, P(FS)=0.25.

Likelihoods P(Exceeds | ·): 0.75 | Exp, 0.20 | Moderate, 0.05 | FS.

SOLUTION

P(Exceeds) = 0.45(0.75) + 0.30(0.20) + 0.25(0.05) = 0.3375 + 0.06 + 0.0125 = **0.41** (41%).

P(Exp | Exceeds) = (0.75 × 0.45) / 0.41 = 0.3375 / 0.41 = **0.8231** (82.31%)

WORKED EXAMPLE **Recession Given Stock Price Increased**

GIVEN

Priors: P(Recession)=0.4, P(Expansion)=0.6. Likelihoods: P(↑ | Rec)=0.3, P(↑ | Exp)=0.8.

SOLUTION

$P(\uparrow) = 0.4(0.3) + 0.6(0.8) = 0.12 + 0.48 = 0.60$

$P(\text{Rec} \mid \uparrow) = 0.3(0.4)/0.60 = 0.12/0.60 = \mathbf{0.20}$ (20%)

Exam Tip: Bayes tree shortcut: (branch joint probability) / (sum of all joint probabilities that produce the observed event).

CHECK YOUR UNDERSTANDING · QUIZ 1

Quiz 1

Select the best answer. Feedback appears after each click.

Q1. EPS probabilities: 0.15/2.60, 0.45/2.45, 0.24/2.20, 0.16/2.00. E(EPS) is closest to:

Q2. Cash flow: Good(0.3, 50), Avg(0.5, 40), Weak(0.2, 20). Expected cash flow =

Q3. P(decline)=0.4; if decline, E(Price)=$97.50; if no decline, E(Price) =$**75.00. Unconditional E(Price) =**

Q4. P(Rec)=0.4, P(Exp)=0.6. P(price ↑ | Rec)=0.3, P(price ↑ | Exp)=0.8. Posterior P(Rec | price ↑) =

Q5. Variance of a random variable is computed using:

LOS B · JOINT, MARGINAL, AND CONDITIONAL PROBABILITIES

Joint, Marginal, and Conditional Probabilities

TYPE	DEFINITION	FORMULA / INTERPRETATION
Unconditional (marginal) P(A)	Probability of event A regardless of other events	Absolute probability
Joint P(AB)	Probability both A and B occur	P(AB) = P(A|B) P(B) = P(B|A) P(A)
Conditional P(A|B)	Probability of A given B has occurred	P(A|B) = P(AB) / P(B)

Multiplication Rule

$$P(A \cap B) = P(A|B)P(B)$$

If A and B are *independent*, P(A | B) = P(A), so P(AB) = P(A) · P(B).

Addition Rule

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If A and B are *mutually exclusive*, P(A ∩ B) = 0, so P(A ∪ B) = P(A) + P(B).

WORKED EXAMPLE

Joint Probability of Two Independent Stocks Rising

GIVEN

P(A rises) = 0.6, P(B rises) = 0.55, independent.

SOLUTION

P(both rise) = 0.6 × 0.55 = **0.33**

P(at least one rises) = 0.6 + 0.55 − 0.33 = **0.82**

LOS A · COVARIANCE AND CORRELATION OF RANDOM VARIABLES

Expected Covariance and Correlation

$$\text{Cov}(R_i, R_j) = \text{E}[R_i - \text{E}(R_i)][R_j - \text{E}(R_j)] = \Sigma\ \text{P}(scenario)\ \cdot\ [R_i - \text{E}(R_i)][R_j - \text{E}(R_j)]$$

$$\rho(R_i, R_j) = \text{Cov}(R_i, R_j)/(\sigma_i\ \cdot\ \sigma_j)$$

Same interpretation as sample correlation: $-1 \le \rho \le +1$; $\rho = 0$ means no linear relation; $\rho = +1$ or -1 denotes perfect linear relations. In a probability model, we weight each scenario by its probability.

Portfolio Expectation Key Result

$$\text{E}(R_p) = \Sigma\ w_i\text{E}(R_i)$$

Expected portfolio return is a simple weighted average — it is *linear*. Variance is NOT linear; it requires covariance.

CHECK YOUR UNDERSTANDING · QUIZ 2

Quiz 2

Select the best answer. Feedback appears after each click.

Q1. A stock has equally likely return outcomes of −10%, +5%, +15%. E(R) and σ(R) are:

Q2. Priors: P(S)=0.45, P(T)=0.30, P(U)=0.25. P(E|S)=0.75, P(E|T)=0.20, P(E|U)=0.05. P(E) equals:

Q3. Continuing above, P(S | E) is closest to:

Q4. If two events A, B are INDEPENDENT, which is TRUE?

Q5. If two events A, B are MUTUALLY EXCLUSIVE, then P(A ∪ B) equals:

LOS B · INVESTMENT APPLICATIONS

Probability Trees in Practice

Common Uses

- **Scenario analysis:** economic states drive conditional EPS/cash-flow distributions.
- **Equity valuation:** combine multi-period dividend forecasts across states.
- **Real options:** value decisions under branching future outcomes.
- **Credit analysis:** default/no-default branches with recovery rates.
- **Bayesian updating:** refine default probabilities given issuer signals (downgrade, missed filing, etc.).

WORKED EXAMPLE **Bayesian Credit Risk**

GIVEN

Prior: P(Default) = 0.03. Likelihood of downgrade: P(DG|D)=0.70, P(DG|ND)=0.05.

SOLUTION

P(DG) = 0.70(0.03) + 0.05(0.97) = 0.021 + 0.0485 = 0.0695

Posterior P(D | DG) = 0.70(0.03)/0.0695 = 0.021/0.0695 = **0.3022** ⇒ default risk revised upward by ~10×.

Exam Tip: Bayes problems are about rewriting likelihoods as fractions of a 2×2 (or 2×n) joint table. Draw the tree, compute every joint cell, take the ratio.

CHECK YOUR UNDERSTANDING · QUIZ 3

Quiz 3

Select the best answer. Feedback appears after each click.

Q1. A $1 binary lottery pays$10 with prob 0.15 and $0 otherwise. Expected value =

Q2. Variance of the same lottery:

Q3. P(A)=0.3, P(B)=0.4, independent. P(A ∪ B) =

Q4. Prior P(D)=0.03; P(DG|D)=0.70, P(DG|ND)=0.05. Updated P(D | DG) is closest to:

MODULE COMPLETE

LM04 · Probability Trees and Conditional Expectations Summary

Final Score

0 / 0

Cheat Sheet

$$\mathrm{E}(X) = \Sigma\ \mathrm{P}(X_i)X_i$$

$$\sigma^2(X) = \Sigma\ \mathrm{P}(X_i)[X_i - \mathrm{E}(X)]^2$$

$$P(A \cap B) = P(A|B)P(B); if independent P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$Total Prob : P(E) = \Sigma\ P(E|S_i)P(S_i)$$

$$Bayes : P(S_n|E) = P(E|S_n)P(S_n)/P(E)$$

Last-Minute Exam Checklist

EPS tree: E(EPS) = 2.3405, σ^2 = 0.038785

Stock-price tree: E(Price)=$84.00

Bayes: P(Exp | Exceeds) = 0.3375/0.41 = **0.8231**

Recession | ↑: 0.12/0.60 = **0.20**

Independence: P(A|B) = P(A)

Step 1 of 11