LM05 · QUANTITATIVE METHODS · CFA LEVEL I

Portfolio Mathematics

Portfolio E(R), variance, covariance, joint-probability model, Roy's safety-first

Core Idea

Portfolio return = weighted average of asset returns. Portfolio variance depends on *every* pairwise covariance — that's what gives diversification its teeth. Safety-first tools rank portfolios by the probability of breaching a shortfall target.

Learning Outcome Statements

LOS a Calculate and interpret the expected value, variance, standard deviation, covariances, and correlations of portfolio returns

LOS b Calculate and interpret the covariance and correlation of portfolio returns using the joint probability function for returns

LOS c Define shortfall risk, calculate the safety-first ratio, and identify an optimal portfolio using Roy's safety-first criterion

Exam Tip: For n assets: n variances + $(n^2-n)/2$ distinct covariances. Five assets → 5 variances and 10 unique covariances.

LOS A · EXPECTED PORTFOLIO RETURN

Expected Return is Linear in Weights

$$E(R_p) = w_1E(R_1) + w_2E(R_2) + \ \dots \ + w_nE(R_n)$$

Each individual $E(R_i)$ is itself a probability-weighted average across scenarios.

WORKED EXAMPLE

Three-Asset Portfolio

GIVEN

S&P 500: w=0.50, E(R)=13%; Corp bonds: w=0.25, E(R)=6%; MSCI EAFE: w=0.25, E(R)=15%.

SOLUTION

$E(R_p)$ = 0.50(13%) + 0.25(6%) + 0.25(15%) = 6.5% + 1.5% + 3.75% = **11.75%**

WORKED EXAMPLE

Scenario-Driven E(R) for A and B

GIVEN

Recession(0.25, A=2%, B=4%); Normal(0.50, A=8%, B=10%); Boom(0.25, A=12%, B=16%). w_A=0.40, w_B=0.60.

SOLUTION

$E(R_A)$ = 0.25(2) + 0.50(8) + 0.25(12) = **7.5%**

$E(R_B)$ = 0.25(4) + 0.50(10) + 0.25(16) = **10.0%**

$E(R_p)$ = 0.40(7.5%) + 0.60(10.0%) = 3.0% + 6.0% = **9.0%**

LOS A · PORTFOLIO VARIANCE

Portfolio Variance with Multiple Assets

Two-Asset Formula

$$\sigma^2(R_p) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\text{Cov}(R_1, R_2)$$

Three-Asset Formula

$$\sigma^2(R_p) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1w_2\text{Cov}(R_1R_2) + 2w_1w_3\text{Cov}(R_1R_3) + 2w_2w_3\text{Cov}(R_2R_3)$$

Historical Covariance

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\text{Cov}(R_{A}, R_{B}) = \Sigma (R_{A,t} - R̄_{A})(R_{B,t} - R̄_{B}) / (n - 1)
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Correlation

$$\rho_{ij} = \text{Cov}(R_i, R_j)/(\sigma_i \cdot \sigma_j) \Rightarrow \text{Cov}(R_i, R_j) = \rho_{ij} \cdot \sigma_i \cdot \sigma_j$$

Counting Parameters

For n securities: **n variances**, **n^2 − n** covariances in a full matrix, **(n^2 − n)/2** unique. For n=5: 5 variances and 10 unique covariances.

Diversification Insight

$\sigma^2(R_p)$ is always positive, but covariances can be positive OR negative. **Adding assets with low or negative covariance lowers portfolio variance**. That is the quantitative mechanism behind diversification.

LOS B · COVARIANCE FROM JOINT PROBABILITY

Joint Probability Function for Returns

$$\text{Cov}(R_{A}, R_{B}) = \Sigma _{i} \Sigma _{j} \text{P}(R_{Ai} \cap R_{Bj}) \; \cdot \; (R_{Ai} - \bar{R}_{A})(R_{Bj} - \bar{R}_{B})$$

Each cell of the joint table contains a joint probability; only the diagonal is non-zero if the scenarios are perfectly paired (e.g., same economic state). **Independence ⇒ $P(R_A R_B) = P(R_A) P(R_B)$**; for independent variables covariance = 0.

WORKED EXAMPLE

Covariance and Correlation of A and B

GIVEN

Scenarios Recession(0.25), Normal(0.50), Boom(0.25). A returns 2%, 8%, 12%; B returns 4%, 10%, 16%.

$\bar{R}_A$=7.5%, $\bar{R}_B$=10%.

σ_A=0.0357, σ_B=0.0424.

SOLUTION

Cov(A, B) = 0.25(2−7.5)(4−10) + 0.50(8−7.5)(10−10) + 0.25(12−7.5)(16−10)

= 0.25(−5.5)(−6) + 0 + 0.25(4.5)(6)

= 8.25 + 0 + 6.75 = 15 (in $\%^2$) ⇒ **Cov = 0.0015** (in decimals)

ρ_{AB} = 0.0015 / (0.0357 × 0.0424) = **0.99**

WORKED EXAMPLE

Portfolio Variance and SD

GIVEN

w_A=0.4, w_B=0.6, σ_A=0.0357, σ_B=0.0424, Cov=0.0015.

SOLUTION

$\sigma^2(R_p) = 0.4^2(0.0357)^2 + 0.6^2(0.0424)^2 + 2(0.4)(0.6)(0.0015)$

= 0.000204 + 0.000647 + 0.000720 = **0.00157**

$\sigma(R_p) = \sqrt{0.00157} = \mathbf{0.0396}$ (3.96%)

CHECK YOUR UNDERSTANDING · QUIZ 1

Quiz 1

Select the best answer. Feedback appears after each click.

Q1. S&P 500 w=0.50 E(R)=13%; Corp bonds w=0.25 E(R)=6%; MSCI EAFE w=0.25 E(R)=15%. $E(R_p)$ =

Q2. For n = 5 assets, the number of UNIQUE covariances is:

Q3. w_A=0.4, w_B=0.6; σ_A=0.0357, σ_B=0.0424; Cov=0.0015. Portfolio SD =

Q4. If R_A and R_B are INDEPENDENT, then:

LOS C · SHORTFALL RISK & SAFETY-FIRST

Roy's Safety-First Criterion

Shortfall Risk

The probability that a portfolio's return (or value) falls **below some minimum acceptable level** R_L over a specified horizon.

$$SFRatio = [\mathrm{E}(R_p) - R_L]/\sigma_p$$

Interprets as *standard deviations above the threshold*. Higher SFRatio ⇒ lower shortfall probability. Optimal portfolio under Roy's criterion: the one with the **highest SFRatio**.

WORKED EXAMPLE **Choosing Between Portfolios**

GIVEN

Target R_L = 2%.

P1: E(R)=8%, σ=10%.

P2: E(R)=12%, σ=15%.

P3: E(R)=10%, σ=12%.

SOLUTION

$SF_1 = (8 - 2)/10 = 0.60$

$SF_2 = (12 - 2)/15 = 0.667$

$SF_3 = (10 - 2)/12 = 0.667$

Both P2 and P3 tie, and dominate P1 ⇒ choose either (real-world: lower SD often preferred on ties).

Normal-Distribution Assumption

If returns are normal: $P(R < R_L) = N(-\text{SFRatio})$. A higher SFRatio translates directly into a lower left-tail probability.

Exam Tip: Roy's criterion is a single-period mean-variance ranking that treats the threshold R_L as the effective risk-free rate. The SFRatio looks algebraically identical to the Sharpe ratio when $R_L = R_f$.

LOS A–C · INTEGRATED APPLICATION

Putting It Together

INTEGRATED EXAMPLE

A 40/60 Equity–Bond Portfolio with Roy's Test

GIVEN

w_A=0.4 (E=7.5%, σ=3.57%); w_B=0.6 (E=10%, σ=4.24%); Cov=0.0015. R_L=3%.

SOLUTION

$E(R_p)$ = 0.4(7.5%) + 0.6(10%) = **9.0%**

σ_p = **3.96%** (computed earlier)

SFRatio = (9 − 3) / 3.96 = **1.515**

Under normality, $P(R_p < 3\%) = N(-1.515) \approx$ **6.5%** shortfall probability.

Takeaways

- Building block: individual $E(R_i)$, σ_i^2, $Cov(R_i, R_j)$.
- Portfolio E(R) is a linear combination of weights and $E(R_i)$.
- Portfolio σ^2 depends on *every* pairwise covariance.
- Joint probability model & historical calculations produce the same covariance.
- Roy's SFRatio = $(E(R_p) - R_L)/\sigma_p$; pick the maximum.

CHECK YOUR UNDERSTANDING · QUIZ 2

Quiz 2

Select the best answer. Feedback appears after each click.

Q1. Scenario table: Rec(0.25,A=2%,B=4%); Norm(0.50,A=8%,B=10%); Boom(0.25,A=12%,B=16%). Cov(A,B) =

Q2. For the same A and B, σ_A=0.0357, σ_B=0.0424, Cov=0.0015. Correlation ρ(A,B) =

Q3. $E(R_p)$ = 12%, σ_p = 15%, R_L = 3%. SFRatio =

Q4. Roy's safety-first criterion selects the portfolio with the:

Q5. For a 3-asset portfolio, total terms in the variance expansion =

CHECK YOUR UNDERSTANDING · QUIZ 3

Quiz 3

Select the best answer. Feedback appears after each click.

Q1. Two assets, equal weights 50/50, $\sigma_A=\sigma_B$=20%, $\rho_{AB}=-1$. Portfolio SD =

Q2. Covariance calculated from a joint probability table differs from historical sample covariance in that it uses:

Q3. If $E(R_p)$=9%, σ_p=3.96%, R_L=3%, SFRatio =

MODULE COMPLETE

LM05 · Portfolio Mathematics Summary

Final Score

0 / 0

Cheat Sheet

$$\mathrm{E}(R_p) = \Sigma\ w_i \mathrm{E}(R_i)$$

$$\sigma^2(R_p) = \Sigma\Sigma\ w_i w_j \mathrm{Cov}(R_i, R_j)$$

$$\mathrm{Cov}(R_A, R_B) = \Sigma\Sigma\ \mathrm{P}(R_{Ai}R_{Bj})(R_{Ai}\bar{-}R_A)(R_{Bj}\bar{-}R_B)$$

$$\rho_{ij} = Cov_{ij}/(\sigma_i\sigma_j)$$

$$SFRatio = [\mathrm{E}(R_p) - R_L]/\sigma_p$$

Last-Minute Exam Checklist

Three-asset $E(R_p)$: 11.75% example

Scenario Cov(A,B): 0.0015, ρ=0.99

Portfolio SD: 3.96% with Cov=0.0015

SFRatio: maximize — higher is safer

n-asset count: $(n^2-n)/2$ unique covariances

Step 1 of 10