LM06 · QUANTITATIVE METHODS · CFA LEVEL I

Simulation Methods

Lognormal distribution, Monte Carlo simulation, bootstrap resampling

Why Simulate?

When an analytical solution is unavailable (path-dependent options, retirement plans with multiple stochastic drivers, sensitivity to multi-factor scenarios), simulation generates a **distribution** of possible outcomes. Lognormal modeling keeps prices positive; Monte Carlo draws future paths from specified distributions; bootstrap resamples from observed history.

Learning Outcome Statements

LOS a Explain the relationship between normal and lognormal distributions and why lognormal is used for asset prices with continuously compounded returns

LOS b Describe Monte Carlo simulation and explain its investment applications

LOS c Describe the use of bootstrap resampling in simulation based on observed data

LOS A · NORMAL VS LOGNORMAL

The Lognormal Distribution

Definition

A variable Y follows a **lognormal** distribution if ln(Y) is normally distributed. Equivalently, if $X \sim N(\mu, \sigma^2)$, then $Y = e^X$ is lognormal.

PROPERTY	NORMAL	LOGNORMAL
Range	$(-\infty, +\infty)$	$(0, +\infty)$ — bounded below by zero
Symmetry	Symmetric	Positively (right) skewed
Parameters	μ, σ^2	Same μ, σ^2 of its associated normal
Use case	Model returns	Model **prices** — prices can't be negative

$$Mean of lognormal: \mu_Y = e^{\mu + 0.5\sigma^2}$$

Why prices are modeled lognormal: asset prices must be positive, and they compound multiplicatively. Equivalently, *continuously compounded returns* are modeled as normal, which forces the price to follow a lognormal distribution.

LOS A · CONTINUOUSLY COMPOUNDED RETURNS

Continuously Compounded Returns and Price Paths

$$S_T/S_0 = 1 + R_H \quad \cdot \quad Discreteholding - periodreturn$$

$$r = ln(S_T/S_0) \quad \cdot \quad Continuouslycompoundedreturn$$

$$S_T = S_0 \ \cdot \ e^{rT}$$

Key property: continuously compounded returns are **additive** across periods.

$$r_{0,T} = r_{0,1} + r_{1,2} + \ \ldots \ + r_{T-1,T}$$

WORKED EXAMPLE

Discrete vs Continuous Return

GIVEN

S_0 = 30, S_T = 34.50.

SOLUTION

R_H = 34.50/30 − 1 = **15.00%** (discrete)

r = ln(34.50/30) = ln(1.15) = **0.13976** (13.976% continuously compounded)

Check: 30 · $e^{0.13976}$ = 34.50 ✓

IID Assumption & Volatility Scaling

If returns are independently and identically distributed (i.i.d.), then r ~ N(μT, σ^2T). Mean and variance scale linearly with time, but standard deviation scales with √T:

$$SDoverTperiods = \sigma \quad T$$

Example: Daily vol = 0.01 (1%). Annualized vol assuming 250 trading days = $0.01 \times \sqrt{250}$ = **15.81%**.

LOS B · MONTE CARLO SIMULATION

Six-Step Monte Carlo Procedure

1. **Specify the quantity of interest** — e.g., MV_p in 10 years.
2. **Specify a time grid** — K sub-periods of length Δt covering the horizon (e.g., 20 six-month steps over 10 years).
3. **Specify distributions** for the key risk factors — e.g., $E(R_p) = 6.4\%$, $\sigma = 12\%$, returns ~ $N(\cdot, \cdot)$.
4. **Draw standard-uniform random numbers** in [0, 1] for each risk factor in each sub-period.
5. **Convert uniform draws to z-values** via the inverse normal CDF. E.g., 0.31561 → −0.48; 0.7673 → +0.73.
6. **Convert z-values to returns and advance portfolio value**. Repeat 1,000+ times → distribution of terminal outcomes.

WORKED EXAMPLE

Single MC Path Increment

GIVEN

$E(R_p)$=6.4%, σ=12%. Random uniform draw 0.31561 → $z = -0.48$. Second draw 0.7673 → $z = +0.73$.

SOLUTION

Period 1: $R = 6.4\% + (-0.48)(12\%) = 6.4\% - 5.76\% = \mathbf{0.64\%}$. $MV_1 = MV_0(1.0064)$.

Period 2: $R = 6.4\% + 0.73(12\%) = 6.4\% + 8.76\% = \mathbf{15.16\%}$. $MV_2 = MV_1(1.0758)$.

Typical Objective

Ex: client wants $MV_{p,10}$ = \$2M with 95% probability-of-success (PoS). After running 1,000 MC paths, rank terminal values; the 5th-percentile path defines $R_{p,95}$. Required beginning capital: $MV_{p,0} = 2M / (1 + R_{p,95})^{10}$.

Limitations of Monte Carlo

- Statistical (not analytical) method — gives estimates, not exact closed-form answers.

- Output only as good as input assumptions.
- Does NOT support cause-and-effect conclusions — correlation is not causation in simulated data either.
- Computationally intensive.

CHECK YOUR UNDERSTANDING · QUIZ 1

Quiz 1

Select the best answer. Feedback appears after each click.

Q1. Which is TRUE of the lognormal distribution?

Q2. S_0 = 30, S_T = 34.50. The continuously compounded return r is closest to:

Q3. If daily volatility is 1% and there are 250 trading days, annualized volatility is closest to:

Q4. A Monte Carlo simulation with 1,000 runs produces:

LOS C · BOOTSTRAP RESAMPLING

Bootstrap — Resampling From Observed Data

Bootstrap Procedure

Resampling repeatedly draws samples from an original observed data sample to estimate population parameters. In **bootstrap resampling**, each draw is made *with replacement*, so the same observation can appear multiple times in a given sample, or not at all.

Steps

1. Start with n observed data points.
2. Draw a sample of size n *with replacement* from the dataset.
3. Compute the statistic of interest (mean, median, percentile, etc.).
4. Repeat steps 2–3 B times (e.g., B = 1,000).
5. Estimate standard error as the SD of the B sample statistics.

FEATURE	MONTE CARLO	BOOTSTRAP

Input distribution	Assumed (e.g., normal, lognormal)	Drawn from observed historical data
Range of outcomes	Not limited by historical data	Limited by the observed sample
Typical use	Path simulations, option pricing	Standard-error estimation, percentile CI

Exam Tip: Bootstrap advantage: no distributional assumption. Limitation: can't produce outcomes outside the observed history. Monte Carlo advantage: can produce tail outcomes beyond history (if the parametric model supports them).

LOS B, C · INVESTMENT APPLICATIONS

Where Simulation Shines

Typical Use Cases

Retirement planning	Probability a portfolio lasts 30 years given withdrawal rate
Risk management (VaR)	Distribution of potential losses over a horizon
Complex securities	Path-dependent options, CDOs, mortgage prepayment models
Capital budgeting	NPV distribution under multi-factor uncertainty
Standard error estimation	Bootstrap the sample statistic to get SE without a closed-form formula

Critical Reminder

Simulation outputs are only as reliable as (1) the parametric assumptions (for MC) or (2) the quality of the observed sample (for bootstrap). A model calibrated to 10 years of benign history will understate a 100-year crisis.

WORKED EXAMPLE

Bootstrap Estimate of Mean Return's Standard Error

GIVEN

60 monthly returns of asset X. Objective: estimate $SE(\bar{R})$.

SOLUTION

1. Draw 60 returns (with replacement) → compute $\bar{R}_b$.
2. Repeat B = 1,000 times → 1,000 sample means.
3. $SE(\bar{R})$ = standard deviation of those 1,000 bootstrapped means.

This avoids assuming normality in R_t.

CHECK YOUR UNDERSTANDING · QUIZ 2

Quiz 2

Select the best answer. Feedback appears after each click.

Q1. Bootstrap resampling draws subsamples:

Q2. Monte Carlo simulation's outputs are:

Q3. Under i.i.d. returns, $\ln(S_T/S_0)$ has variance equal to:

Q4. Bootstrap CANNOT produce outcomes:

Q5. Why is the lognormal distribution commonly used for asset prices?

CHECK YOUR UNDERSTANDING · QUIZ 3

Quiz 3

Select the best answer. Feedback appears after each click.

Q1. A Monte Carlo simulation for retirement modeling runs 1,000 10-year paths. The 50th path ends at $1.80M. This represents approximately:

Q2. A continuously compounded return of 10% corresponds to a discrete holding-period return of:

Q3. Which of the following is a limitation of Monte Carlo simulation?

MODULE COMPLETE

LM06 · Simulation Methods Summary

Final Score

0 / 0

Cheat Sheet

$$r = ln(S_T/S_0); S_T = S_0 e^{rT}$$

$$Ifr\ N(\mu T, \sigma^2 T), SD = \sigma \quad T$$

$$Y = e^X \ \Rightarrow\ Y\ Lognormalif X\ N(\mu, \sigma^2)$$

$$MeanoflognormalY : \mu_Y = e^{\mu\ +0.5\sigma^2}$$

$$MC : specify\ \rightarrow\ grid\ \rightarrow\ distributions\ \rightarrow\ draw\ \rightarrow\ z\ \rightarrow\ path\ \rightarrow\ repeat$$

$$Bootstrap : with - replacementresamplingfromobserveddata$$

Last-Minute Exam Checklist

Continuous return: ln(34.50/30)=0.13976

Vol scaling: daily 1% → annual 15.81% (√250)

MC increment: 6.4% + z×12% per step

Monte Carlo: statistical, assumption-dependent

Bootstrap: with-replacement from observed data

Lognormal: positive, right-skewed, for asset prices

Step 1 of 10